Centre d'Etude et de Recherche en Economie,
Gestion, Modélisation et Informatique Appliquée

Document de travail 2010-06

Septembre 2010

"Well posedness in multiparametric algebras"

Antoine DELCROIX, Victor DEVOUE & Jean-André MARTI

Université des Antilles et de la Guyane. Faculté de Droit et d'Economie de la Martinique. Campus de Schoelcher - Martinique FWI
B.P. 7209 - 97275 Schoelcher Cedex - Tél. : 0596. 72.74.00 - Fax. : 0596. 72.74.03

Well posedness in multiparametric Algebras

Antoine Delcroix
Equipe Analyse Algébrique Non Linéaire
Laboratoire AOC
Faculté des sciences exactes et naturelles
Université des Antilles et de la Guyane
97159 Pointe à Pitre Cedex, Guadeloupe. F.W.I.

Victor Dévoué et Jean-André Marti
Equipe Analyse Algébrique Non Linéaire
Laboratoire CEREGMIA
Campus de Shoelcher B.P. 7209
Université des Antilles et de la Guyane
97275 Schoelcher Cedex Martinique F.W.I.

Abstract

By means of convenient regularizations for an ill posed Cauchy problem, we define an associated generalized problem and discuss the conditions for the solvability of it. To illustrate this, starting from the semilinear unidirectional wave equation with non Lipschitz nonlinearity and irregular data given on a characteristic curve, we show existence and uniqueness of the solution.

Contents

Mathematics Subject Classification (2010): 35A01, 35A02, 35A25, 35D99, 46F30, 46T30

Keywords: regularization of data, regularization of problems, distributions, nonlinear generalized functions, nonlinear problems, characteristic problems, wave equation.

1 Introduction

Solving partial differential equations can meet some obstructions, as nonlinearities, irregular coefficients and/or data, characteristic cases. This generally leads to an ill-posed problem in classical frameworks (Sobolev spaces, distributions theory,...). Our aim is to show that, given a classical ill posed problem, we can define an associated generalized problem, in an adapted ($\mathcal{C}$,$\mathcal{E}$,$\mathcal{P}$)-algebra [9, 19, 20, 21] by means of suitable regularizations. We shall discuss its solvability within this algebra and discuss its well posedness in the Hadamard sense. Note that, when several obstacles to the existence of classical solution are mixed in the same problem, the framework of ($\mathcal{C}$,$\mathcal{E}$,$\mathcal{P}$)-algebras allow more flexibility in the choice of the regularizations, than simplified Colombeau generalized functions [3, 15, 22, 23]. (Each regularization can be taken in charge by its own regularizing parameter.) We refer the reader to [16, 25] in which similar ideas of regularizing non linear problems within algebras of generalized functions are used.

The paper is organized as follows. In section 2, for sake of self contentedness, we recall the construction of ($\mathcal{C}$,$\mathcal{E}$,$\mathcal{P}$)-algebras and more recent advances related to the generation of the ring $\mathcal{C}$ of generalized constants. In section 3, we introduce the tools needed to construct the generalized problem associated to the classical ill posed one, namely generalized operators associated to a stability property of the ($\mathcal{C}$,$\mathcal{E}$,$\mathcal{P}$)-algebra (which are used to regularize the right hand side of the equation) and generalized restriction operators (which are used to de-characterize the problem). After describing the different regularization procedures, we give an existence theorem of the solution to the generalized problem and we study the uniqueness of the solution. The well posedness of the problem and the independence of the solution with respect to the regularizations are discussed in Subsection 4.4. However, obtaining general results in this direction is still a work in progress. We illustrate the use of this tools, in Subsection 4.7, with a characteristic Cauchy problem with irregular data and non Lipschitz nonlinearity in the right hand side, based on the transport equation.

2 The presheaf of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-type algebras

2.1 Algebraic and topological structures

We begin by recalling the notions from [19, 20] that form the basis for our study.

Definition 1 *(a) Let:*
(1) Λ *be a set of indices;*
(2) A *be a solid subring of the ring* $\mathbb{K}^\Lambda$*, ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), that is: Whenever* $(|s_\lambda|)_\lambda \leq (|r_\lambda|)_\lambda$ *(i.e.: For any λ, we have $|s_\lambda| \leq |r_\lambda|$) for some $((s_\lambda)_\lambda, (r_\lambda)_\lambda) \in \mathbb{K}^\Lambda \times A$, it follows that: $(s_\lambda)_\lambda \in A$;*
(3) I_A *be a solid ideal of* A*;*
(4) $\mathcal{E}$ *be a sheaf of $\mathbb{K}$-topological algebras over a topological space X;*
Moreover, suppose that
(5) *For any open set Ω in X, the algebra $\mathcal{E}(\Omega)$ is endowed with a family $\mathcal{P}(\Omega) = (P_i)_{i\in I(\Omega)}$ of semi-norms such that if Ω_1, Ω_2 are two open subsets of X with $\Omega_1 \subset \Omega_2$, it follows that $I(\Omega_1) \subset I(\Omega_2)$ and if ρ_1^2 is the restriction operator $\mathcal{E}(\Omega_2) \to \mathcal{E}(\Omega_1)$, then, for each $P_i \in \mathcal{P}(\Omega_1)$ the semi-norm $\widetilde{P}_i = P_i \circ \rho_1^2$ extends P_i to $\mathcal{P}(\Omega_2)$;*
(6) *Let $\Theta = (\Omega_h)_{h\in H}$ be any family of open set in X with $\Omega = \cup_{h\in H}\Omega_h$. Then, for each $P \in \mathcal{P}(\Omega)$, there exist a finite subfamily $\Omega_1, \ldots, \Omega_{n(i)}$ of Θ and corresponding semi-norms $P_1 \in \mathcal{P}(\Omega_1), \ldots, P_{n(i)} \in \mathcal{P}(\Omega_{n(i)})$ such that, for any $u \in \mathcal{E}(\Omega)$,*

$$P(u) \leq P_1(u\,|_{\Omega_1}) + \ldots + P_{n(i)}\left(u\,|_{\Omega_{n(i)}}\right).$$

(b) Define $|B| = \{(|r_\lambda|)_\lambda,\ (r_\lambda)_\lambda \in B\}$, $B = A$ or I_A, and set

$$\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega) = \{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda \mid \forall i \in I(\Omega), ((P_i(u_\lambda))_\lambda \in |A|\},$$
$$\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(\Omega) = \left\{(u_\lambda)_\lambda \in [\mathcal{E}(\Omega)]^\Lambda \mid \forall i \in I(\Omega), (P_i(u_\lambda))_\lambda \in |I_A|\right\},$$
$$\mathcal{C} = A/I_A.$$

Note that, from (2), $|A|$ is a subset of A and that $A_+ = \{(b_\lambda)_\lambda \in A, \forall\lambda \in \Lambda,\ b_\lambda \geq 0\} = |A|$. The same holds for I_A. Furthermore, (2) implies also that A is a $\mathbb{K}$-algebra.

Proposition 1 *[19, 20]*
(i) $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ (resp. $\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$) is a sheaf of $\mathbb{K}$-subalgebras (resp. of ideals) of the sheaf $\mathcal{E}^\Lambda$ (resp. of $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$).
(ii) The factor $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$ is a presheaf with localization principle in addition.

With the assumption (2), the constant sheaf $\mathcal{H}_{(A,\mathbb{K},|.|)}/\mathcal{J}_{(I_A,\mathbb{K},|.|)}$ is exactly the sheaf $\mathcal{C} = A/I_A$.

Definition 2 *We call presheaf of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra, the factor presheaf of algebras over the ring $\mathcal{C} = A/I_A$*

$$\mathcal{A} = \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$$

and we denote by $[u_\lambda]$ the class in $\mathcal{A}(\Omega)$ defined by $(u_\lambda)_{\lambda\in\Lambda} \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$.

Remark 1 *If A is a ring with unity, the map $\iota : \mathbb{K} \to A$ defined by $\iota(r) = (r)_\lambda$ is an embedding of algebra and induces a morphism of ring from $\mathbb{K} \to \mathcal{C}$ (Lemma 14, [20]). Moreover, if Λ is left-filtering for a given (partial) order relation $\prec$ and if*

$$I_A \subset \{(a_\lambda)_\lambda \in A \mid \lim_\Lambda a_\lambda = 0\}, \tag{1}$$

then, the morphism ι is injective.

Example 1 ***Relationship with distribution theory and Colombeau algebras***

One main feature of this construction is that we can choose the triple $(\mathcal{C}, \mathcal{E}, \mathcal{P})$ such that the sheaves C^∞ *and* $\mathcal{D}'$ *are embedded in the corresponding sheaf* $\mathcal{A}$. *In particular, we can multiply (the images of) distributions in* $\mathcal{A}$.

We consider the sheaf $\mathcal{E} = \mathrm{C}^\infty$ *over* $\mathbb{R}^d$, *where* $\mathcal{P}$ *is the usual family of topologies* $(\mathcal{P}_\Omega)_{\Omega \in \mathcal{O}(\mathbb{R}^d)}$. *Here* $\mathcal{O}\left(\mathbb{R}^d\right)$ *denotes the set of all open sets of* $\mathbb{R}^d$. *Let us recall that* $\mathcal{P}_\Omega$ *is defined by the family of semi-norms* $(p_{K,l})_{K \Subset \Omega, l \in \mathbb{N}}$ *with*

$$\forall f \in \mathrm{C}^\infty(\Omega), \quad p_{K,l}(f) = \sup_{x \in K, |\alpha| \leq l} |D^\alpha f(x)|. \tag{2}$$

(The notation $K \Subset \mathbb{R}^d$, *used in the sequel, means that the set* K *is a compact set included in* $\mathbb{R}^d$.*) From Lemma 14 in [19], it follows that the canonical maps, defined for any* $\Omega \in \mathcal{O}\left(\mathbb{R}^d\right)$ *by*

$$\sigma_\Omega : \mathrm{C}^\infty(\Omega) \to \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega), \quad f \mapsto (f)_\lambda$$

are injective morphism of algebras if, and only if, A *is unitary. Under this assumption, these maps give rise to a canonical sheaf embedding of* C^∞ *into* $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}$ *and (using a partition of unity in* C^∞ *inducing a sheaf structure on* $\mathcal{A}$*) to a canonical sheaf morphism of algebras from* C^∞ *into* $\mathcal{A}$. *This sheaf morphism turns out to be a sheaf morphism of embeddings if* Λ *is a directed set with respect to a partial order* $\prec$ *and if relation (1) holds.*

We shall address the question of the embedding of $\mathcal{D}'$ *for the simple case of* $\Lambda = (0,1]$. *For a net of mollifiers* $(\varphi_\varepsilon)_\varepsilon$ *given by*

$$\varphi_\varepsilon(x) = \left(1/\varepsilon^d\right) \varphi(x/\varepsilon), \; x \in \mathbb{R}^d \quad \textit{where } \varphi \in \mathcal{D}(\mathbb{R}^d) \textit{ and } \textstyle\int \varphi(x)\, dx = 1,$$

and $T \in \mathcal{D}'\left(\mathbb{R}^d\right)$, *the net* $(T * \varphi_\varepsilon)_\varepsilon$ *is a net of smooth functions in* $\mathrm{C}^\infty\left(\mathbb{R}^d\right)$, *moderately increasing in* $1/\varepsilon$. *This means that*

$$\forall K \Subset \mathbb{R}^d, \forall l \in N, \quad \exists m \in \mathbb{N} : \; p_{K,l}\; (T * \varphi_\varepsilon) = \mathrm{o}(\varepsilon^{-m}), \; \textit{as } \varepsilon \to 0. \tag{3}$$

This justifies to choose

$$A = \left\{ (r_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]} \mid \exists m \in \mathbb{N} \; : \; |u_\varepsilon| = \mathrm{o}(\varepsilon^{-m}), \; \textit{as } \varepsilon \to 0 \right\}$$
$$I = \left\{ (r_\varepsilon)_\varepsilon \in \mathbb{R}^{(0,1]} \mid \forall q \in \mathbb{N} \; : \; |u_\varepsilon| = \mathrm{o}(\varepsilon^{q}), \; \textit{as } \varepsilon \to 0 \right\}.$$

In this case (with $\mathcal{E} = \mathrm{C}^\infty$*), the sheaf of algebras* $\mathcal{A} = \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}/\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$ *is exactly the so-called special Colombeau algebra* $\mathcal{G}$ *[2, 15, 23]. Then, for all* $\Omega \in \mathcal{O}\left(\mathbb{R}^d\right)$, $\mathrm{C}^\infty(\Omega)$ *is embedded in* $\mathcal{A}(\Omega)$ *by*

$$\sigma_\Omega : \mathrm{C}^\infty(\Omega) \to \mathcal{A}(\Omega) \qquad f \mapsto [f_\varepsilon] \; \textit{ with } f_\varepsilon = f \textit{ for all } \varepsilon \textit{ in } (0,1],$$

because the constant net $(f)_\varepsilon$ *belongs to* $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}\left(\mathbb{R}^d\right)$ *and* $(f)_\varepsilon \in \mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$ *implies* $f = 0$ *in* $\mathrm{C}^\infty(\Omega)$. *Furthermore,* $\mathcal{D}'\left(\mathbb{R}^d\right)$ *is embedded in* $\mathcal{A}\left(\mathbb{R}^d\right)$ *by the mapping*

$$\iota : T \mapsto (T * \varphi_\varepsilon)_\varepsilon$$

Indeed, relation (3) implies that $(T * \varphi_\varepsilon)_\varepsilon$ *belongs to* $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})}\left(\mathbb{R}^d\right)$ *and* $(T * \varphi_\varepsilon)_\varepsilon \in \mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}$ *implies that* $T * \varphi_\varepsilon \to 0$ *in* $\mathcal{D}'\left(\mathbb{R}^d\right)$, *as* $\varepsilon \to 0$ *and* $T = 0$. *Thus,* ι *is a well defined injective map.*

With the help of cutoff functions, we can define analogously, for each open set Ω *in* $\mathbb{R}^d$, *an embedding* ι_Ω *of* $\mathcal{D}'(\Omega)$ *into* $\mathcal{A}(\Omega)$, *and finally a sheaf embedding* $\mathcal{D}' \to \mathcal{A}$. *This embedding depends on the choice of the net of mollifiers* $(\varphi_\varepsilon)_\varepsilon$. *We refer the reader to [8, 22] for more complete discussions about embeddings in Colombeau's case and to [19] for the case of* $(\mathcal{C}, \mathcal{E}, \mathcal{P})$*-algebras.*

Definition 3 *Tempered generalized functions, [15], [27], [28]. For $f \in \mathrm{C}^{\infty}(\mathbb{R}^n)$, $r \in \mathbb{Z}$ and $m \in \mathbb{N}$, we put*

$$\mu_{r,m}(f) = \sup_{x\in\mathbb{R}^n, |\alpha|\leq m} (1+|x|)^r \left|\mathcal{D}^{\alpha} f(x)\right|.$$

The space of functions with slow growth is

$$\mathcal{O}_M(\mathbb{R}^n) = \{f \in \mathrm{C}^{\infty}(\mathbb{R}^n) : \forall m \in \mathbb{N}, \exists q \in \mathbb{N}, \mu_{-q,m}(f) < +\infty\}.$$

Definition 4 *We put*

$$\mathcal{X}_{\mathcal{T}}(\mathbb{R}^n) = \{(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^n)^{(0,1]} : \forall m \in \mathbb{N},\ \exists q \in \mathbb{N}, \exists N \in \mathbb{N}, \mu_{-q,m}(f_\varepsilon) = O(\varepsilon^{-N}) \quad (\varepsilon \to 0)\},$$
$$\mathcal{N}_{\mathcal{T}}(\mathbb{R}^n) = \{(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^n)^{(0,1]} : \forall m \in \mathbb{N},\ \exists q \in \mathbb{N}, \forall p \in \mathbb{N}, \mu_{-q,m}(f_\varepsilon) = O(\varepsilon^{p}) \quad (\varepsilon \to 0)\}.$$

$\mathcal{X}_{\mathcal{T}}(\mathbb{R}^n)$ is a subalgebra of $\mathcal{O}_M(\mathbb{R}^n)^{(0,1]}$ and $\mathcal{N}_{\mathcal{T}}(\mathbb{R}^n)$ an ideal of $\mathcal{X}_{\mathcal{T}}(\mathbb{R}^n)$. The algebra $\mathcal{G}_{\mathcal{T}}(\mathbb{R}^n) = \mathcal{X}_{\mathcal{T}}(\mathbb{R}^n) / \mathcal{N}_{\mathcal{T}}(\mathbb{R}^n)$ is called the algebra of tempered generalized functions. The generalized derivation $\mathcal{D}^{\alpha} : u = [u_\varepsilon] \mapsto \mathcal{D}^{\alpha} u = [\mathcal{D}^{\alpha} u_\varepsilon]$ provides $\mathcal{G}_{\mathcal{T}}(\mathbb{R}^n)$ with a differential algebraic structure.

We return to the general case with the assumption that A is a ring with unity and Λ is left-filtering for the given (partial) order relation $\prec$. In practice, the ring A and the ideal I_A are constructed as described below.

Definition 5 ***(Overgenerated rings)***. *Choose B_p a finite family of p nets in $(\mathbb{R}_+^*)^{\Lambda}$ (usually given by the asymptotic structure of the problem.) Consider B the subset of elements in $(\mathbb{R}_+^*)^{\Lambda}$ obtained as rational fractions with coefficients in $\mathbb{R}_+^*$, of elements in B_p as variables. Define*

$$A = \left\{(a_\lambda)_\lambda \in \mathbb{K}^{\Lambda} \mid \exists\, (b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leq b_\lambda\right\}.$$

We say that A is overgenerated *by B_p (and it is easy to see that A is a solid subring of $\mathbb{K}^{\Lambda}$).If I_A is some solid ideal of A, we also say that $\mathcal{C} = A/I_A$ is* overgenerated *by B_p.*
For example, as a "canonical" ideal of A, we can take

$$I_A = \left\{(a_\lambda)_\lambda \in \mathbb{K}^{\Lambda} \mid \forall\, (b_\lambda)_\lambda \in B, \exists \lambda_0 \in \Lambda, \forall \lambda \prec \lambda_0 : |a_\lambda| \leq b_\lambda\right\}.$$

Remark 2 *This definition implies that B is stable by inverse.*

2.2 An association process

Let us denote by:

- Ω an open subset of X,
- $\mathcal{F}$ a given sheaf of topological $\mathbb{K}$-vector spaces (resp. $\mathbb{K}$-algebras) over X containing $\mathcal{E}$ as a subsheaf,
- a a map from $\mathbb{R}_+$ to A_+ such that $a(0) = 1$ (for $r \in \mathbb{R}_+$, we denote $a(r)$ by $(a_\lambda(r))_\lambda$).

For $(v_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$, we shall denote by $\lim_{\Lambda,\mathcal{F}(\Omega)} v_\lambda$ the limit of $(v_\lambda)_\lambda$ for the $\mathcal{F}$-topology when it exists. We recall that $\lim_{\Lambda,\mathcal{F}(\Omega)} u_\lambda|_V = f \in \mathcal{F}(V)$ iff, for each $\mathcal{F}$-neighborhood W of f, there exists $\lambda_0 \in \Lambda$ such that

$$\lambda \prec \lambda_0 \Longrightarrow a_\lambda(r)\, u_\lambda \in W.$$

We suppose also that we have, for each open subset $V \subset \Omega$,

$$\mathcal{J}_{(I_A,\mathcal{E},\mathcal{P})}(V) \subset \left\{(v_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(V) : \lim_{\Lambda,\mathcal{F}(\Omega)} v_\lambda = 0\right\}. \tag{4}$$

Definition 6 *Consider $u = [u_\lambda] \in \mathcal{A}(\Omega)$, $r \in \mathbb{R}_+$, V an open subset of Ω and $f \in \mathcal{F}(V)$. We say that u is a* (r)-associated to f in V*:*

$$u \underset{\mathcal{F}(V)}{\overset{a(r)}{\sim}} f$$

if $\lim_{\Lambda,\mathcal{F}(\Omega)} \left(a_\lambda\left(r\right) u_\lambda\left|_V\right.\right) = f$.
In particular, if $r = 0$, u and f are said associated *in V.*

Example 2 *Take $X = \mathbb{R}^d$, $\mathcal{F} = \mathcal{D}'$, $\Lambda =]0,1]$, $\mathcal{A} = \mathcal{G}$, $V = \Omega$, $r = 0$. The usual association between $u = [u_\varepsilon] \in \mathcal{G}\left(\Omega\right)$ and $T \in \mathcal{D}'\left(\Omega\right)$ is defined by*

$$u \sim T \Longleftrightarrow u \underset{\mathcal{D}'(\Omega)}{\overset{a(0)}{\sim}} T \Longleftrightarrow \lim_{\varepsilon\to 0,\mathcal{D}'(\Omega)} u_\varepsilon = T.$$

3 General Framework, generalized stability and restriction

3.1 General framework

As recalled above, the $(\mathcal{C},\mathcal{E},\mathcal{P})$ algebras are constructed by means of three independent "parameters": $\mathcal{C},\mathcal{E},\mathcal{P}$. Like in Example 1, we shall use throughout the paper $\mathcal{E} = \mathrm{C}^\infty$ with $X = \mathbb{R}^d$ and $d = 1$ or 2, $\mathcal{P}\left(\mathbb{R}^d\right)$ being the usual family of semi norms $\left(P_{K,l}\right)_{K\Subset\mathbb{R}^d, l\in\mathbb{N}}$ defined by relation (2). We shall construct latter the asymptotic structure given by $\mathcal{C} = A/I_A$, in relationship with the regularizations of the ill posed problem. However, for any choice of $\mathcal{C}$, we recall that $\mathcal{A}$ is a sheaf of differential algebras with $D^\alpha u = [D^\alpha u_\lambda]$ where $(u_\lambda)_{\lambda\in\Lambda}$ is any representative of $u \in \mathcal{A}\left(\mathbb{R}^d\right)$, Λ being the set of indices given in the previous definition of $\mathcal{A}$. In this context we set $\mathcal{H}_{(A,\mathcal{E},\mathcal{P})} = \mathcal{H}$ and $\mathcal{J}_{(A,\mathcal{E},\mathcal{P})} = \mathcal{J}$. For sake of simplicity, we keep the same sheaf symbols $\mathcal{H}$, $\mathcal{J}$, $\mathcal{A} = \mathcal{H}/\mathcal{J}$ when $X = \mathbb{R}$ or $\mathbb{R}^2$ when writing $\mathcal{H}\left(\mathbb{R}\right)$ (resp. $\mathcal{J}\left(\mathbb{R}\right)$, resp. $\mathcal{A}\left(\mathbb{R}\right)$) or $\mathcal{H}\left(\mathbb{R}^2\right)$ (resp. $\mathcal{J}\left(\mathbb{R}^2\right)$, resp. $\mathcal{A}\left(\mathbb{R}^2\right)$) which are two distinct algebras constructed on the same ring $\mathcal{C} = A/I_A$.

Under some technical hypotheses on the set B, which are satisfied for the algebras considered in this paper, we have the analogue of theorem 1.2.3. of [15] for $(\mathcal{C},\mathcal{E},\mathcal{P})$-algebras. We suppose here that Λ is left filtering and give this proposition for $\mathcal{A}\left(\mathbb{R}^2\right)$, although it is valid in more general situations. This result will be used in Section 4.3.

Proposition 2 *Assume that the set B, introduced in Definition 5, is stable by inverse and that there exists $(a_\lambda)_\lambda \in B$ with $\lim_\Lambda a_\lambda = 0$. Consider $(u_\lambda)_\lambda \in \mathcal{H}(\mathbb{R}^2)$ such that*

$$\forall K \Subset \mathbb{R}^2, \quad \left(P_{K,0}\left(u_\lambda\right)\right)_\lambda \in \left|I_A\right|.$$

Then $(u_\lambda)_\lambda \in \mathcal{J}(\mathbb{R}^2)$.

Proof. Take $K \Subset \Omega$. We have to prove that $\forall l \in \mathbb{N}$, $P_{K,l}\left(u_\lambda\right) \in |I_A|$. By induction, it suffices to prove that $P_{K,0}\left(u_\lambda\right) \in |I_A|$ implies $P_{K,1}\left(u_\lambda\right) \in |I_A|$. In fact, this amounts to show that $P_{K,0}\left(u_\lambda\right) \in |I_A|$ implies $P_{K,0}\left(\left(\partial/\partial x_i\right) u_\lambda\right) \in |I_A|$ for $i \in \{1,\ldots,d\}$. Set $\delta = \min(1, \mathrm{dist}(K, \partial\Omega))$ and $L = K + \overline{B}(0,\delta/2)$. We have $K \Subset L \Subset \Omega$.

Since $(u_\lambda)_\lambda \in \mathcal{H}_{(A,\mathcal{E},\mathcal{P})}(\Omega)$, there exists $(\beta_\lambda)_\lambda \in B$ such that

$$\exists\lambda_0 \in \Lambda, \forall\lambda \prec \lambda_0, \quad P_{L,2}\left(u_\lambda\right) \leq \beta_\lambda.$$

We may assume that $\lim_\Lambda \beta_\lambda = +\infty$. Indeed, for any $(\beta_\lambda)_\lambda \in B$, we set $\beta'_\lambda = \max\left(a_\lambda^{-1}, \beta_\lambda\right)$ where $(a_\lambda)_\lambda \in B$ is such that $\lim_\Lambda a_\lambda = 0$. Thus, $\lim_\Lambda \max \beta'_\lambda = +\infty$.

Take any $(c_\lambda)_\lambda \in B$ and define $b_\lambda = \dfrac{a_\lambda c_\lambda}{a_\lambda + c_\lambda}$. Clearly we have $b_\lambda \in |A|$, $b_\lambda \le c_\lambda$ and $b_\lambda \le a_\lambda$, then $\lim_\Lambda b_\lambda = 0$. Let e_i, $i = 1, 2$, the canonical base of $\mathbb{R}^2$. There exists λ_1 such that, for all $x \in K$, $x + b_\lambda \beta_\lambda^{-1} e_i \in L$ when $\lambda \prec \lambda_1$, since $\lim_\Lambda \beta_\lambda^{-1} = 0$. By the Taylor theorem, we have, for $x \in K$,

$$u_\lambda\left(x + b_\lambda \beta_\lambda^{-1} e_i\right) = u_\lambda(x) + b_\lambda \beta_\lambda^{-1} \frac{\partial}{\partial x_i} u_\lambda(x) + \frac{1}{2}\left(b_\lambda \beta_\lambda^{-1}\right)^2 \frac{\partial^2}{\partial x_i^2} u_\lambda\left(x + \theta(b_\lambda \beta_\lambda^{-1} e_i)\right)$$

with $0 \le \theta \le 1$. It follows that

$$\frac{\partial}{\partial x_i} u_\lambda(x) = b_\lambda^{-1} \beta_\lambda \left(u_\lambda\left(x + b_\lambda \beta_\lambda^{-1} e_i\right) - u_\lambda(x)\right) - \frac{1}{2}\left(b_\lambda \beta_\lambda^{-1}\right) \frac{\partial^2}{\partial x_i^2} u_\lambda\left(x + \theta(b_\lambda \beta_\lambda^{-1} e_i)\right).$$

Thus

$$\left|\frac{\partial}{\partial x_i} u_\lambda(x)\right| \le 2 b_\lambda^{-1} \beta_\lambda P_{L,0}(u_\lambda) + (1/2)\, b_\lambda \beta_\lambda^{-1} P_{L,2}(u_\lambda) \le 2 b_\lambda^{-1} \beta_\lambda P_{L,0}(u_\lambda) + \frac{1}{2} b_\lambda$$

for $\lambda \prec \min(\lambda_0, \lambda_1)$.

As $P_{K,0}(u_\lambda) \in |I_A|$, we have $P_{L,0}(u_\lambda) \le (1/4) b_\lambda^2 \beta_\lambda^{-1} \in B$, for $\lambda \prec \lambda_2$ for some λ_2. Thus

$$\left|\frac{\partial}{\partial x_i} u_\lambda(x)\right| \le b_\lambda, \quad \text{for } \lambda \prec \min(\lambda_0, \lambda_1, \lambda_2).$$

Thus $P_{K,0}((\partial/\partial x_i) u_\lambda) \le b_\lambda \,(\le c_\lambda)$, for $\lambda \prec \min(\lambda_0, \lambda_1, \lambda_2)$ and $P_{K,0}((\partial/\partial x_i) u_\lambda) \in |I_A|$. ■

3.2 Generalized operator associated to a stability property

Consider another set of indices M, a map $\mu : \Lambda \to M$, and a family $(F_\mu)_{\mu \in M}$ of functions in $\mathrm{C}^\infty\left(\mathbb{R}^3\right)$. Set $f \in \mathrm{C}^\infty\left(\mathbb{R}^2\right)$, we define

$$\begin{aligned} \mathrm{C}^\infty\left(\mathbb{R}^2\right) &\mapsto \mathrm{C}^\infty\left(\mathbb{R}^2\right) \\ f &\mapsto G_\lambda(f) = F_{\mu(\lambda)}(.,.,f), \end{aligned}$$

$$F_{\mu(\lambda)}(.,.,f) : \ (t,x) \mapsto F_{\mu(\lambda)}(t,x,f(t,x)).$$

Clearly, the family $(G_\lambda)_\lambda$ maps $\left(\mathrm{C}^\infty\left(\mathbb{R}^2\right)\right)^\Lambda$ into $\left(\mathrm{C}^\infty\left(\mathbb{R}^2\right)\right)^\Lambda$ and allows to define a map from $\mathcal{A}\left(\mathbb{R}^2\right)$ into $\mathcal{A}\left(\mathbb{R}^2\right)$ as follows.

Definition 7 *We say that the algebra $\mathcal{A}\left(\mathbb{R}^2\right)$ if* stable under $\left(F_{\mu(\lambda)}\right)_{\lambda \in \Lambda}$ *(or $(G_\lambda(f))_\lambda$) if:*
(i) $\forall (u_\lambda)_\lambda \in \mathcal{H}(\mathbb{R}^2)$, $F_{\mu(\lambda)}(.,.,u_\lambda) \in \mathcal{H}(\mathbb{R}^2)$,
(ii) $\forall (i_\lambda)_\lambda \in \mathcal{J}(\mathbb{R}^2)$, $F_{\mu(\lambda)}(.,.,u_\lambda + i_\lambda) - F_{\mu(\lambda)}(.,.,u_\lambda) \in \mathcal{J}(\mathbb{R}^2)$.

Under the conditions (i) and (ii), for $u = [u_\lambda] \in \mathcal{A}\left(\mathbb{R}^2\right)$, $\left(\left[F_{\mu(\lambda)}(.,.,u_\lambda)\right]\right)$ is a well defined element of $\mathcal{A}\left(\mathbb{R}^2\right)$ (i.e. not depending on the representative $(u_\lambda)_\lambda$ of u). This leads to the following:

Definition 8 *If $\mathcal{A}\left(\mathbb{R}^2\right)$ if stable under $\left(F_{\mu(\lambda)}\right)_{\lambda \in \Lambda}$, the operator*

$$\mathcal{F} : \mathcal{A}\left(\mathbb{R}^2\right) \to \mathcal{A}\left(\mathbb{R}^2\right), \quad u = [u_\lambda] \mapsto \left[F_{\mu(\lambda)}(.,.,u_\lambda)\right] = \left[G_\lambda(u_\lambda)\right]$$

is called the generalized operator associated to the family $\left(F_{\mu(\lambda)}\right)_{\lambda \in \Lambda}$.

Remark 3 *In the sequel we shall use a more handable notion involving explicitly the family of seminorms* $(P_{K,l})_{K\Subset\mathbb{R}^d,l\in\mathbb{N}}$. *We say that the family* $\left(F_{\mu(\lambda)}\right)_{\lambda\in\Lambda}$ *is* $\mathcal{A}\left(\mathbb{R}^2\right)$-moderate *if*

(i') *For each* $K\Subset\mathbb{R}^2$, *for each* $l\in\mathbb{N}$, *for each* $(u_\lambda)_\lambda\in\left(\mathrm{C}^\infty(\mathbb{R}^2)\right)^\Lambda$, *there is a positive finite sequence* $(C_{j,\lambda})_{j=0,,l}$ *with* $(C_{j,\lambda})_\lambda\in|A|$ *such that*

$$P_{K,l}(F_{\mu(\lambda)}(.,.,u_\lambda))\leq\sum\nolimits_{j=0}^{l}C_{j,\lambda}P_{K,l}^{j}(u_\lambda).$$

If (i') *holds,* (i) *clearly holds too. Thus, if* (i') *and* (ii) *hold, the corresponding generalized operator* $\mathcal{F}$ *is well defined.*

3.3 Generalized restriction mapping

Consider another set of indices N, a map $\nu:\Lambda\to N$, and a family $(\Phi_\nu)_{\nu\in N}$ of functions in $\mathrm{C}^\infty(\mathbb{R})$. Set $f\in\mathrm{C}^\infty\left(\mathbb{R}^2\right)$, we define

$$\begin{aligned}\mathrm{C}^\infty(\mathbb{R})&\to\mathrm{C}^\infty(\mathbb{R})\\ f&\mapsto R_\lambda(f),\end{aligned}$$

$$R_\lambda(f):t\mapsto f\left(t,\Phi_{\nu(\lambda)}(t)\right)).$$

The family $(R_\lambda)_\lambda$ maps $\left(\mathrm{C}^\infty\left(\mathbb{R}^2\right)\right)^\Lambda$ into $\left(\mathrm{C}^\infty(\mathbb{R})\right)^\Lambda$.

Definition 9 *The family of smooth functions* $\left(t\mapsto\Phi_{\nu(\lambda)}(t)\right)_{\lambda\in\Lambda}$ *is* compatible with second side restriction *(of any* $u\in\mathcal{A}\left(\mathbb{R}^2\right)$ *towards some* $v\in\mathcal{A}(\mathbb{R})$ *if*

$$\forall\,(u_\lambda)_\lambda\in\mathcal{H}(\mathbb{R}^2),\quad\left(u_\lambda\left(.,\Phi_{\nu(\lambda)}(.)\right)\right)_\lambda\in\mathcal{H}(\mathbb{R})\ ;\quad\forall\,(i_\lambda)_\lambda\in\mathcal{J}(\mathbb{R}^2),\quad\left(i_\lambda\left(.,\Phi_{\nu(\lambda)}(.)\right)\right)_\lambda\in\mathcal{J}(\mathbb{R}).$$

Clearly, if $u=[u_\lambda]\in\mathcal{A}(\mathbb{R}^2)$ then $\left[u_\lambda\left(.,\Phi_{\nu(\lambda)}(.)\right]\right.$ is a well defined element of$\mathcal{A}(\mathbb{R})$ (i.e. not depending on the representative of u.) This leads to the following:

Definition 10 *If the family* $\left(t\mapsto\Phi_{\nu(\lambda)}(t)\right)_{\lambda\in\Lambda}$ *is compatible with second side restriction, the mapping*

$$\mathcal{R}:\mathcal{A}\left(\mathbb{R}^2\right)\to\mathcal{A}(\mathbb{R}),\quad u=[u_\lambda]\mapsto\left[u_\lambda\left(.,\Phi_{\nu(\lambda)}(.)\right]\right.=\left[R_\lambda\left(u_\lambda\right)\right]$$

is called the generalized second side restriction mapping *associated to the family* $\left(\Phi_{\nu(\lambda)}\right)_{\lambda\in\Lambda}$.

Remark 4 *The previous process generalizes the standard one defining the restriction of the generalized function* $u=[u_\lambda]\in\mathcal{A}\left(\mathbb{R}^2\right)$ *to the manifold* $\{x=\Phi(t)\}$ *obtained when taking* $\Phi_{\nu(\lambda)}=\Phi$ *for each* $\lambda\in\Lambda$.

In the following proposition we give sufficient conditions, which generalize the c-boundeness of [15], for the compatibility with second side restriction.

Proposition 3 *Assume that:*
(i) *For each* $K\Subset\mathbb{R}$, *there exists* $K'\Subset\mathbb{R}$ *such that, for all* $\lambda\in\Lambda$, $\Phi_{\nu(\lambda)}(K)\subset K'$,
(ii) $\left(\Phi_{\nu(\lambda)}\right)_\lambda$ *belongs to* $\mathcal{H}(\mathbb{R})$.
Then the family $\left(t\mapsto\Phi_{\nu(\lambda)}(t)\right)_{\lambda\in\Lambda}$ *is compatible with second side restriction.*

Proof. Take $(u_\lambda)_\lambda$ (resp. $(i_\lambda)_\lambda$) in $\mathcal{H}(\mathbb{R}^2)$ (resp. $\mathcal{J}(\mathbb{R}^2)$) and set $v_\lambda(t) = u_\lambda\left(t, \Phi_{\nu(\lambda)}(t)\right.$. From (i) we have

$$\begin{aligned} &p_{K,0}(v_\lambda) \leq p_{K\times K',0}(u_\lambda) \\ &p_{K,1}(v_\lambda) \leq p_{K\times K',(1,0)}(u_\lambda) + p_{K\times K',(0,1)}(u_\lambda)\, p_{K,1}\left(\Phi_{\nu(\lambda)}\right). \end{aligned}$$

By induction we can see that for each $K \Subset \mathbb{R}$, and each $l \in \mathbb{N}$, $p_{K,l}(v_\lambda)$ is estimated by sums or products of terms like $p_{K\times K',(n,m)}(u_\lambda)$ for $n+m \leq l$, or $p_{K,k}\left(\Phi_{\nu(\lambda)}\right)$ for $k \leq l$. Then, from (ii), $p_{K,l}(v_\lambda)$ is in $|A|$. Similarly, setting $j_\lambda(t) = i_\lambda\left(t, \Phi_{\nu(\lambda)}(t)\right.$ leads to $p_{K,l}(j_\lambda) \in |I_A|$. Then $\left(u_\lambda\left(., \Phi_{\nu(\lambda)}(.)\right)\right)_\lambda$ (resp. $i_\lambda\left(., \Phi_{\nu(\lambda)}(.)\right)$) belongs to $\mathcal{H}(\mathbb{R})$ (resp. $\mathcal{J}(\mathbb{R})$). ■

3.4 A generalized differential problem associated to a ill posed classical one

Our goal is to give a meaning to the differential Cauchy problem formally written as

$$(P_{form})\left\{\begin{array}{c} P(D)u = F(.,.,u) \\ u\mid_\gamma = f \end{array}\right.$$

where $P(D)$ is a linear differential hyperbolic of order one, F a nonlinear function of its arguments may be non Lipschitz, γ a manifold may be characteristic, f a datum may be as irregular as a distribution. We don't have a classical surrounding in which we can pose (and a fortiori solve) the problem. In the sequel, by means of regularizing processes we will define an associated problem to (P_{form})

$$(P_{gen})\left\{\begin{array}{c} P(D)u = \mathcal{F}(u) \\ \mathcal{R}(u) = \mathbf{f} \end{array}\right.$$

where u is searched in some convenient algebra $\mathcal{A}\left(\mathbb{R}^2\right)$, $\mathcal{F}$ and $\mathcal{R}$ are defined as previously, $\mathbf{f} = \left[f_{\zeta(\lambda)}\right]$ being some given element in $\mathcal{A}(\mathbb{R})$ constructed from f. More precisely, ζ is a map $\Lambda \to Z$ where Z is a third set of indices, and $\mathbf{f} = \left(f_{\zeta(\lambda)}\right)_\lambda + \mathcal{J}(\mathbb{R})$.

In terms of representatives, and thanks to the stability and second side restriction hypothesis, solving (P_{gen}) amounts to find a family $(u_\lambda)_\lambda \in \mathcal{H}(\mathbb{R}^2)$ such that

$$(P_{rep})\left\{\begin{array}{c} P(D)u_\lambda(t,x) - F_{\mu(\lambda)}(t,x,u_\lambda(t,x)) = i_\lambda(t,x) \\ u_\lambda\left(t, \Phi_{\nu(\lambda)}(t)\right. - f_{\xi(\lambda)}(t) = j_\lambda(t) \end{array}\right.$$

where $(i_\lambda)_\lambda \in \mathcal{J}\left(\mathbb{R}^2\right)$ and $(j_\lambda)_\lambda \in \mathcal{J}(\mathbb{R})$. Suppose we can find $u_\lambda \in \mathrm{C}^\infty\left(\mathbb{R}^2\right)$ verifying

$$(P_{infty})\left\{\begin{array}{c} P(D)u_\lambda(t,x) = F_{\mu(\lambda)}(t,x,u_\lambda(t,x)) \\ u_\lambda\left(t, \Phi_{\nu(\lambda)}(t)\right. = f_{\xi(\lambda)}(t) \end{array}\right.$$

then, if we can prove that $(u_\lambda)_\lambda \in \mathcal{H}(\mathbb{R}^2)$, $u = [u_\lambda]$ is a solution of (P_{gen}).

We have to prove that the generalized solution of (P_{gen}) only depend on the class of cut off functions.

4 Application to a characteristic Cauchy problem with irregular data and non Lipschitz nonlinearity

We treat the characteristic Cauchy problem for the transport equation formally written in characteristic coordinates

$$(P_c)\ \left\{\begin{array}{c} \dfrac{\partial u}{\partial t} = F(.,.,u) \\ u\mid_{\{t=0\}} = T \end{array}\right.$$

where F is non Lipschitz and $T \in \mathcal{D}'(\mathbb{R})$.

4.1 From the ill posed problem (P_c) to a well posed formulation (P_g)

4.1.1 First step: definition of three regularizing processes

- **Regularization of the non Lipschitz nonlinearity: $\mathbf{H_{Lip}}$ hypothesis**

a) We start from $F \in \mathrm{C}^\infty\left(\mathbb{R}^3,\mathbb{R}\right)$. Let $(r_\eta)_\eta \in \mathbb{R}_+^{(0,1]}$ be such that $r_\eta > 0$ and $\lim_{\eta\to 0} r_\eta = +\infty$. Consider a family of smooth one-variable functions $(g_\eta)_\eta$ such that $0 \leq g_\eta \leq 1$ and

$$g_\eta(z) = 0 \text{ if } |z| \geq r_\eta \ ; \quad g_\eta(z) = 1 \text{ if } -r_\eta + 1 \leq z \leq r_\eta - 1.$$

Moreover assume that, for every integer $n > 0$, $g_\eta^{(n)}$ is bounded independently of η and set

$$\sup_{z\in[-r_\eta;r_\eta]} \left| g_\eta^{(n)}(z) \right| = M_n.$$

b) Set $\phi_\eta(z) = zg_\eta(z)$. We approximate the function $(t,x,z) \mapsto F(t,x,z)$ by the family of functions

$$((t,x,z) \mapsto F_\eta(t,x,z))_\eta = (F(t,x,\phi_\eta(z)))_\eta \,. \tag{5}$$

All the derivatives $D^\alpha F_\eta(t,x,z)$ are bounded for all z when (t,x) lies in a compact set of $\mathbb{R}^2$; particularly, for each $l \in \mathbb{N}$ we set

$$M_{K,\eta,l} = \sup_{(t,x)\in K,\ z\in\mathbb{R},\ |\alpha|\leq l} \left| D^\alpha F_\eta(t,x,z) \right|.$$

- **Regularization of characteristic situation: $\mathbf{H_{car}}$ hypothesis**

a) We approximate the characteristic curve $\{t = 0\}$ by the family of non characteristic ones $\{x = l_\varepsilon(t)\}_{\varepsilon\in(0,1]}$ where l_ε is a smooth function with strictly positive derivative and image $\mathbb{R}$. We suppose that the family $(l_\varepsilon)_\varepsilon$ tends simply to 0 (or uniformly on each compact which is equivalent here) when ε tends to 0.
b) We assume that

$$\text{For each } K \Subset \mathbb{R} \text{ there exists } K' \Subset \mathbb{R} \text{ such that, for all } \varepsilon \in (0,1] : l_\varepsilon(K) \subset K'.$$

- **Regularization of data: $\mathbf{H_{data}}$ hypothesis**

Consider $\varphi \in \mathcal{D}(\mathbb{R})$ such that $\int \varphi(t)\,\mathrm{d}t = 1$. Take φ_ρ, defined by $\varphi_\rho(t) = (1/\rho)\,\varphi(1/\rho)$, as mollifier. We define the family $(f_\rho)_{\varepsilon,\eta,\rho}$ by

$$f_\rho = \varphi_\rho * T.$$

4.1.2 Second step: H_1-hypothesis and generalized formulation

The first relationship between the construction of $\mathcal{C} = A/I_A$ and the previous regularizations is given by the following hypothesis

$$(H_1) \quad \begin{cases} (i)\ \text{For each } K \Subset \mathbb{R}^2 \text{ and each } l \in \mathbb{N}, \text{ the family } \left(M_{K,\eta,l}\right)_{\varepsilon,\eta,\rho} \text{ lies in } |A| \\ (ii)\ \text{For each } K \Subset \mathbb{R}, \text{ and each } l \in \mathbb{N} \text{ the family } \left(P_{K,l}(l_\varepsilon)\right)_{\epsilon,\eta,\rho} \text{ belongs to } |A| \\ (iii)\ \text{For each } K \Subset \mathbb{R}, \text{ and each } l \in \mathbb{N} \text{ the family } \left(P_{K,l}(f_\rho)\right)_{\epsilon,\eta,\rho} \text{ belongs to } |A|\,. \end{cases}$$

Now we can associate to (P_c) the generalized problem (P_g)

$$(P_g)\left\{\begin{array}{l}\dfrac{\partial u}{\partial t}=\mathcal{F}(u)\\ \mathcal{R}\,(u)=\mathbf{f}\end{array}\right.$$

where $\mathcal{F}$, the generalized operator associated to the stability, $\mathcal{R}$ the generalized restriction mapping and $\mathbf{f}=[f_\rho]$ are studied below.

- **Generalized operator associated to the stability**

Here the set of indices is $\Lambda=(0,1]^3$ and λ denoted by (ε,η,ρ). Moreover $\mu\,(\lambda)$ is chosen as η. We have to prove first that $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under $(F_\eta)_{\varepsilon,\eta,\rho}$. We shall use the following lemmas.

Lemma 4 *Set $H\in\mathrm{C}^\infty(\mathbb{R}^3,\mathbb{R})$, $u\in\mathrm{C}^\infty(\mathbb{R}^2,\mathbb{R})$, $G(t,x)=H(t,x,u(t,x))$. For any $n\geq 0$, $m\geq 0$, $\alpha=(n,m)$, with $|\alpha|=n+m>1$, then*

$$\frac{\partial^{n+m}G}{\partial x^n\partial^m y}(t,x)=\sum_{|\beta|\leq n+m}c_\beta D^\beta H(t,x,u(t,x))\sum_{i=1}^{n+m}\sum_{p_i(\alpha,\beta)}d_{i,\alpha,\beta}\prod_{j=1}^{i}\left(D^{l_j}u(t,x)\right)^{k_j}+$$
$$\frac{\partial H}{\partial z}(t,x,u(t,x))D^{(n,m)}u(t,x)\ ,$$

where $\beta\in\mathbb{N}^3$, $c_\beta\geq 0$ and $c_{(0,0,1)}=0$, $d_{i,\alpha,\beta}\geq 0$. The set $p_i(\alpha,\beta)$ mentioned in the inner sum consists of all nonzero multi-indices $(k_1,...,k_i,l_1,...,l_i)\in\mathbb{N}^i\times\mathbb{N}^{2i}$, such that

$$0\prec l_1\prec...\prec l_i,\quad\sum_{j=1}^{i}k_j=|\beta|\,,\quad\sum_{j=1}^{i}k_jl_j=\alpha,$$

the linear order on $\mathbb{N}/^2$ is defined by: if $\mu=(\mu_1,\mu_2)$ and $\nu=(\nu_1,\nu_2)$ are in $\mathbb{N}^2$, $\mu\prec\nu$ provided of the following holds

$$\left\{\begin{array}{c}|\mu|<|\nu|\,;\\ |\mu|=|\nu|\ \text{ and }\mu_1<\nu_1\ \text{ or }\ |\mu|=|\nu|\,,\ \mu_1=\nu_1\text{and }\mu_2<\nu_2.\end{array}\right.$$

The proof uses the Multivariate Faà di Bruno's formula [4]. This lemma implies the:

Lemma 5 *Assume that all the derivatives $D^\beta H(t,x,z)$ are bounded when (t,x) lies in the compact sets of $\mathbb{R}^2$ for all $z\in\mathbb{R}$. Set*

$$k_{K,\beta}=\sup_{(t,x)\in K;\ z\in\mathbb{R}}\left|D^\beta H(t,x,z)\right|,\quad m_K=\sup_{(t,x)\in K;\ z\in\mathbb{R}}\left|\frac{\partial}{\partial z}H(t,x,z)\right|,$$

then there exists a constant $C_{(n,m)}$, independent of H and u, such that, for any $K\Subset\mathbb{R}^2$

$$\left|\frac{\partial^{n+m}G}{\partial t^n\partial^m x}(t,x)\right|\leq C_{(n,m)}\sum_{|\beta|\leq n+m}c_\beta k_{K,\beta}\left(P_{K,n+m}(u)\right)^{|\beta|}+m_K\left|D^{n+m}u(t,x)\right|$$

Remark 5 *In the sequel we suppose that all the regularizations are given by the corresponding hypothesis and that (H_1) is fulfilled. The parameter λ is taken as the triple (ε,η,ρ) but for sake of simplicity, we keep the symbol λ in the following proof.*

Theorem 6 *The algebra $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under the family $(F_\eta)_{\varepsilon,\eta,\rho}$.*

Proof. We remark that we have: $\forall\,(t,x)\in\mathbb{R}^2, F_\eta(t,x,0)=0$. According to the previous lemmas, we obtain

$$\sup_{(t,x)\in K}\left|D^{n+m}F_\eta(.,.,u_\lambda)\right|\leq C_{(n,m)}\sum_{|\beta|\leq n+m}c_\beta M_{K,\eta,|\beta|}\left(P_{K,n+m}(u_\lambda)\right)^{|\beta|}+M_{K,\eta,1}P_{K,n+m}(u_\lambda)$$

for any $K\Subset\mathbb{R}^2$. Hence

$$(6)\qquad P_{K,n+m}\left(F_\eta(.,.,u_\lambda)\right)\leq C_{(n,m)}\sum_{|\beta|\leq n+m}c_\beta M_{K,\eta,|\beta|}\left(P_{K,n+m}(u_\lambda)\right)^{|\beta|}+M_{K,\eta,1}P_{K,n+m}(u_\lambda),$$

where the coefficients $C_{(n,m)}$, $c_\beta M_{K,\eta,|\beta|}$ and $M_{K,\eta,1}$ don't depend on $(u_\lambda)_\lambda$. Thus, according to the part (i) of H_1 hypothesis verified by the family $\left(M_{K,\eta,|\beta|}\right)_{\varepsilon,\eta,\rho}$, the property (i') in Remark 3 is satisfied and consequently the property (i) in Definition 7.

For $(v_\lambda)_\lambda\in\mathcal{H}(\mathbb{R}^2)$, $(w_\lambda)_\lambda\in\mathcal{J}(\mathbb{R}^2)$, we have

$$\begin{aligned}\Delta_\lambda(t,x)&=F_\eta(t,x,v_\lambda(t,x)+w_\lambda\,(t,x))-F_\eta(t,x,v_\lambda(t,x))\\&=w_\lambda\,(t,x)\underbrace{\int_0^1\frac{\partial F_\eta}{\partial z}(t,x,v_\lambda(t,x)+\sigma w_\lambda(t,x))\,\mathrm{d}\sigma}_{\Psi_\lambda(t,x)}.\end{aligned}$$

From relation (6) applied with $u_\lambda=v_\lambda(t,x)+\sigma w_\lambda(t,x)$, we get that $(\Psi_\lambda)_\lambda\in\mathcal{H}(\mathbb{R}^2)$. As $(w_\lambda)_\lambda\in\mathcal{J}(\mathbb{R}^2)$, the property (ii) in Definition 7 is satisfied. Thus $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under $(F_\eta)_{\varepsilon,\eta,\rho}$. ■

Corollary 7 *Take $F\,(t,x,z)=L(z)=z^p$, $p\in\mathbb{N}^*$, and suppose that the family $(r_\eta^p)_{\varepsilon,\eta,\rho}$ lies in $|A|$. (Here we have: $L_\eta(z)=F_\eta(t,x,z)$.). Then $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under $(L_\eta)_{\varepsilon,\eta,\rho}$.*

Proof. We have $|L_\eta(z)|=|z^pg_\eta^p(z)|\leq r_\eta^p$, so $\sup_{(t,x)\in\mathbb{R};z\in\mathbb{R}}|L_\eta(z)|\leq r_\eta^p$. As $\phi_\eta(z)=zg_\eta(z)$, we obtain

$$\frac{\partial^n\phi_\eta}{\partial z^n}(z)=z\frac{\partial^n g_\eta}{\partial z^n}(z)+n\frac{\partial^{n-1}g_\eta}{\partial z^{n-1}}(z).$$

Thus $\left|\dfrac{\partial^n\phi_\eta}{\partial z^n}(z)\right|\leq r_\eta M_n+nM_{n-1}\leq\alpha_n r_\eta$, where $\alpha_n=2\max(M_n;nM_{n-1})$. Set $w(z)=z^p$, then $\dfrac{\partial^m w}{\partial z^m}(z)=\left(\prod_{i=0}^{i=m-1}(p-i)\right)z^{p-m}$ for $1\leq m\leq p$. According to Faà di Bruno's formula, the n^{th} order derivative of $G_\eta=w\circ\phi_\eta$ can be written

$$\frac{\partial^n L_\eta}{\partial z^n}=\sum_{m=1}^{n}\sum_{\substack{i_1\geq\ldots\geq i_m\\ i_1+\ldots+i_m=n}}t_{i_1,\ldots,i_m}w^{(m)}\circ\phi_\eta\prod_{k=1}^{m}\phi_\varepsilon^{(i_k)},$$

where the coefficients $t_{i_1,\ldots,i_m}$ are integers. Then we get

$$\left|\frac{d^n L_\eta}{dz^n}(z)\right|\leq\sum_{m=1}^{p}\sum_{\substack{i_1\geq\ldots\geq i_m\\ i_1+\ldots+i_m=n}}t_{i_1,\ldots,i_m}\left(\prod_{i=0}^{i=m-1}(p-i)\right)r_\eta^{p-m}\prod_{k=1}^{m}\alpha_{i_k}r_\eta\leq\mu_n r_\eta^p,$$

where μ_n is a positive constant depending only upon n. The part (i) of H_1 hypothesis are verified and Theorem 6 gives the result. ■

Thanks to the previous $\mathrm{H_{Lip}}$ hypothesis and the part (i) of H_1 hypothesis leading to the stability of $\mathcal{A}\left(\mathbb{R}^2\right)$ under $(F_\eta)_{(\varepsilon,\eta,\rho)\in(0,1]^3}$, the corresponding operator $\mathcal{F}:\mathcal{A}\left(\mathbb{R}^2\right)\to\mathcal{A}\left(\mathbb{R}^2\right)$ is well defined for $u=[u_{\varepsilon,\eta,\rho}]$ by

$$\mathcal{F}\,(u):=[F_\eta(.,.,u_{\varepsilon,\eta,\rho})]$$

according to Definition 8.

- **Generalized restriction mapping**

Thanks to the previous $\mathrm{H}_{\mathrm{car}}$ hypothesis, and the part (ii) of the hypothesis (H_1), the generalized restriction operator $\mathcal{R} : \mathcal{A}\left(\mathbb{R}^2\right) \to \mathcal{A}\left(\mathbb{R}\right)$ is well defined for $u = \left[u_{\varepsilon,\eta,\rho}\right]_{\mathcal{A}(\mathbb{R}^2)}$ by

$$\mathcal{R}\left(u\right) := \left[t \mapsto u_{\varepsilon,\eta,\rho}\left(t, l_\varepsilon\left(t\right)\right)\right]_{\mathcal{A}(\mathbb{R})}$$

according to Definition 10.

- **Regularization of data**

Proposition 8 *If the family $(\rho)_{\varepsilon,\eta,\rho}$ is contained in the set of generators of A, the family $(f_\rho)_{\varepsilon,\eta,\rho}$ is the representative of an element* $\mathbf{f}$ *in $\mathcal{A}\left(\mathbb{R}\right)$.*

Proof. From $\mathrm{H}_{\mathrm{data}}$ hypothesis, one can see that

$$\forall K \Subset \mathbb{R},\ \exists b \in \mathbb{R}_+,\ \forall l \in \mathbb{N},\ p_{K,l}\left(f_\rho\right) = O\left(\rho^{-l-b}\right) \text{ as } \rho \to 0. \tag{7}$$

This follows from the local structure of distributions: on each compact K, T can be written as finite sum of terms as $D^\beta f$, where f is a continuous function and then, $\sup_{x\in K}\left|D^\alpha\left(D^\beta f * \varphi_\rho\right)(x)\right| = O\left(\rho^{-|\alpha|-|\beta|}\right)$. It follows that

$$\forall K \Subset \mathbb{R},\ \forall l \in \mathbb{N},\ \exists N \in \mathbb{N},\ p_{K,l}\left(f_\rho\right) = O\left(\rho^{-N}\right) \text{ as } \rho \mapsto 0.$$

Consequently, if $(\rho)_{\varepsilon,\eta,\rho}$ is a generator of A, we deduce from the part (iii) of the hypothesis (H_1) that $\left(f_\rho\right)_{e,\eta,\rho}$ is the representative of a generalized function $\mathbf{f} \in \mathcal{A}\left(\mathbb{R}\right)$. ∎

4.2 The regularized problem (P_∞) and H_2 hypothesis

As exposed in Subsection 3.4, in order to solve (P_g) we begin to solve the regularized problem

$$(P_\infty)\begin{cases} \dfrac{\partial}{\partial t}\left(u_{\varepsilon,\eta,\rho}\left(t,x\right)\right) = F_\eta(t,x,u_{\varepsilon,\eta,\rho}\left(t,x\right)) \\ u_{\varepsilon,\eta,\rho}\left(t, l_\varepsilon(t)\right) = f_\rho\left(t\right) \end{cases}$$

in $\mathrm{C}^\infty\left(\mathbb{R}^2\right)$.

- **Remarks, notations and hypothesis.**

Each compact $K \Subset \mathbb{R}^2$ is contained in some product $[-a,a]\times[-b,b]$. Set $\beta_{K,\varepsilon} = \max(a, l_\varepsilon^{-1}(b))$ and $\alpha_{K,\varepsilon} = \min(-a, l_\varepsilon^{-1}(-b))$. Define $a_{K,\varepsilon} = 2\max(\beta_{K,\varepsilon}, |\alpha_{K,\varepsilon}|)$ and $K_\varepsilon = K_{1\varepsilon}\times K_2$ with $K_{1\varepsilon} = [-a_{K,\varepsilon}/2, a_{K,\varepsilon}/2]$ and $K_2 = [-b,b] = [-c/2, c/2]$. By construction we have $K \subset K_\varepsilon$.

If we take $K = K_\varepsilon$ in the hypothesis on f_ρ given in (7) we obtain

$$\forall\varepsilon\in(0,1]\,,\forall K \Subset \mathbb{R}^2, \exists Q_{K,\varepsilon}\in\mathbb{N}^*,\ \forall\beta\in\mathbb{N},\ \exists D_\beta\in\mathbb{R}_+^*,\ \sup_{t\in K_{1\varepsilon}}\left|D^\beta f_\rho(t)\right| \le D_\beta\rho^{-\beta}\rho^{-Q_{K,\varepsilon}} \tag{8}$$

We shall assume that

$$(H2)\quad\begin{cases} \forall\left(\varepsilon,\eta\right)\in(0,1]^2, \forall K\Subset\mathbb{R}^2, \forall l\in\mathbb{N}, \exists\mu_{K,l}>0, \exists M_{\varepsilon,\eta}>0, \\ \sup\limits_{(t,x,z)\in K_\varepsilon\times\mathbb{R},\ |\alpha|\le l}\left|D^\alpha F_\eta(t,x,z)\right| = M_{K,\varepsilon,\eta,l} \le \mu_{K,l}M_{\varepsilon,\eta}. \\ \forall\varepsilon\in(0,1], \forall K\Subset\mathbb{R}^2, \exists\nu_K>0, \exists a_\varepsilon>0, a_{K,\varepsilon}\le\nu_K a_\varepsilon. \\ \forall\varepsilon\in(0,1], \forall K\Subset\mathbb{R}^2, \exists\omega_K>0, \exists Q_\varepsilon>0, Q_{K,\varepsilon}\le\omega_K Q_\varepsilon. \end{cases}$$

Particularly, we set

$$m_{K,\varepsilon,\eta} = \sup_{(t,x)\in K_\varepsilon;\ z\in\mathbb{R}}\left|\frac{\partial}{\partial z}F_\eta(t,x,z)\right| \le \mu_{K,1}M_{\varepsilon,\eta}.$$

Proposition 9 *With the previous hypothesis, the problem* (P_∞) *admits a unique smooth solution* $u_{\varepsilon,\eta,\rho}$ *such that*

$$u_{\varepsilon,\eta,\rho}(t,x) = f_\rho(l_\varepsilon^{-1}(x)) + \int_{l_\varepsilon^{-1}(x)}^{t} F_\eta(\tau, x, u_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau. \tag{9}$$

Moreover

$$\|u_{\varepsilon,\eta,\rho}\|_{\infty,K} \leq \Phi_{K,\varepsilon,\eta,\rho} \exp(m_{K,\varepsilon,\eta} a_{K,\varepsilon}).$$

where $\Phi_{K,\varepsilon,\eta,\rho} = D_0 \rho^{-Q_{K,\varepsilon}} + a_{K,\varepsilon} M_{K\varepsilon,\eta,l}$. *Moreover we have the estimate*

$$\|u_{\varepsilon,\eta,\rho}\|_{\infty,K} \leq \left(A_K \rho^{-Q_\varepsilon} + B_K a_\varepsilon M_{\varepsilon,\eta}\right) \left(\exp a_\varepsilon M_{\varepsilon,\eta}\right)^{C_K}$$

where the constant $A_K = D_0 \omega_K, B_K = \mu_{K,0}\nu_K, C_K = \mu_{K,1}\nu_K$ *depend only upon the compact* K.

Proof. From classical results (for example from the Cauchy-Lipschitz theorem applied for fixed x), we obtain that the problem (P_∞) admits a unique solution $u_{\varepsilon,\eta,\rho}$ defined on $\mathbb{R}^2$, since $F_\eta(t,x,z)$ is bounded on $K_\varepsilon \times \mathbb{R}$. The solution $u_{\varepsilon,\eta,\rho}$ satisfies obviously (9), from which, it can be easily proved by induction that $u_{\varepsilon,\eta,\rho}$ belongs to $\mathrm{C}^\infty(\mathbb{R}^2)$. We have, for $(\tau,x) \in \mathbb{R}^2$,

$$F_\eta(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x)) = F_\eta(\tau,x,0) + u_{\varepsilon,\eta,\rho}(\tau,x) \int_0^1 \frac{\partial F_\eta}{\partial z}(\tau,x,\sigma u_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\sigma.$$

For $(t,x) \in K$, we have $l_\varepsilon^{-1}(x) \in K_{1\varepsilon}$ and for $\tau \in \left[l_\varepsilon^{-1}(x), t\right]$, we also have $\tau \in K_{1\varepsilon}$. Thus,

$$\begin{aligned}
|F_\eta(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x))| &\leq |F_\eta(\tau,x,0)| + |u_{\varepsilon,\eta,\rho}(\tau,x)| \int_0^1 \left|\frac{\partial F_\eta}{\partial z}(\tau,x,\sigma u_{\varepsilon,\eta,\rho}(\tau,x))\right| \mathrm{d}\sigma \\
&\leq M_{K,\varepsilon,\eta,0} + |u_{\varepsilon,\eta,\rho}(\tau,x)|\, m_{K,\varepsilon,\eta}.
\end{aligned}$$

Replacing in (9), we get

$$\begin{aligned}
|u_{\varepsilon,\eta,\rho}(t,x)| &\leq \left|f_\rho(l_\varepsilon^{-1}(x))\right| + \left|t - l_\varepsilon^{-1}(x)\right| M_{K,\varepsilon,\eta,0} + \int_{l_\varepsilon^{-1}(x)}^{t} |u_{\varepsilon,\eta,\rho}(\tau,x)|\, m_{K,\varepsilon,\eta}\,\mathrm{d}\tau \\
&\leq D_0 \rho^{-Q_{K,\varepsilon}} + a_{K,\varepsilon} M_{K,\varepsilon,\eta,0} + \int_{l_\varepsilon^{-1}(x)}^{t} |u_{\varepsilon,\eta,\rho}(\tau,x)|\, m_{K,\varepsilon,\eta}\,\mathrm{d}\tau.
\end{aligned}$$

Using the Gronwall lemma, we get

$$\begin{aligned}
|u_{\varepsilon,\eta,\rho}(t,x)| &\leq \left(D_0 \rho^{-Q_{K,\varepsilon}} + a_{K,\varepsilon} M_{K,\varepsilon,\eta,0}\right) \exp\left|t - l_\varepsilon^{-1}(x)\right| m_{K,\varepsilon,\eta} \\
&\leq \left(D_0 \rho^{-Q_{K,\varepsilon}} + a_{K,\varepsilon} M_{K,\varepsilon,\eta,0}\right) \exp a_{K,\varepsilon} m_{K,\varepsilon,\eta}.
\end{aligned}$$

According Hypothesis $(H2)$, we obtain the estimate

$$|u_{\varepsilon,\eta,\rho}(t,x)| \leq \left(D_0 \omega_K \rho^{-Q_\varepsilon} + \mu_{K,0}\nu_K a_\varepsilon M_{\varepsilon,\eta}\right) \exp \mu_{K,1}\nu_K a_\varepsilon M_{\varepsilon,\eta}$$

■

4.3 The generalized problem (P_g)

4.3.1 Final hypothesis and asymptotic structure of the algebra $\mathcal{A}\left(\mathbb{R}^d\right)$, $d = 1, 2$

To solve the generalized problem (P_g) we need some informations on the derivatives of l_ε^{-1} and add to the previous hypotheses the last following one

$$(H_3) \quad \forall\varepsilon \in (0,1], \forall K \Subset \mathbb{R}^2, \exists\xi_{K,l} \geq 0, \exists P_\varepsilon > 0, \quad \sup_{x\in K_2,\ k\leq l} \left| \left(l_\varepsilon^{-1}\right)^{(k)}(x) \right| = p_{K_2,l}\left(l_\varepsilon^{-1}\right) \leq \xi_{K,l} P_\varepsilon^l$$

We can now built the $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-type algebra $\mathcal{A}\left(\mathbb{R}^d\right)$ associated to the problem. We recall that $\mathcal{A}$ is constructed on $\mathcal{E} = \mathrm{C}^\infty$ with $X = \mathbb{R}^d$ and $d = 1$ or 2, $\mathcal{P}\left(\mathbb{R}^d\right)$ being the usual family of semi norms $(P_{K,l})_{K\Subset\mathbb{R}^d, l\in\mathbb{N}}$ given by

$$P_{K,l}(f) = \sup_{x\in K,\ |\alpha|\leq l} \left| D^\alpha f(x) \right|.$$

The philosophy of our algebras is to adapt the asymptotic structure (defined by A and I_A) to the singularities of the problem "broken" by the three regularizations. The interactions of these regularizations can be see in the progressive construction of the asymptotic structure: collecting all the previous hypotheses from (H_1) to (H_3), we have to suppose that A is overgenerated by the families $\left(M_{\varepsilon,\eta}\right)_{\varepsilon,\eta,\rho}$, $\left(\rho^{Q_\varepsilon}\right)_{\varepsilon,\eta,\rho}$, $\left(P_\varepsilon\right)_{\varepsilon,\eta,\rho}$, $(\exp(M_{\varepsilon,\eta} a_\varepsilon))_{\varepsilon,\eta,\rho}$, and I_A canonically associated to A as specified in Definition 5.

If $\mathcal{C} = A/I_A$ is choose exactly in that way, we can see easily that Assumptions (H_2) and (H_3) imply the more general one (H_1) which was needed to get the generalized formulation (P_g). (H_2) was needed to solve (P_∞), $(H_2)$and (H_3) will be needed to solve (P_g). Then we can collect in only one final formulation (H) the sufficient conditions to solve our problem.

$$(H) \qquad \begin{cases} (i)\ \forall(\varepsilon,\eta) \in (0,1]^2, \forall K \Subset \mathbb{R}^2, \forall l \in \mathbb{N}, \exists\mu_{K,l} > 0, \exists M_{\varepsilon,\eta} > 0, \\ \qquad \sup\limits_{(t,x,z)\in K_\varepsilon\times\mathbb{R},\ |\alpha|\leq l} \left| D^\alpha F_\eta(t,x,z) \right| = M_{K_\varepsilon,\eta,l} \leq \mu_{K,l} M_{\varepsilon,\eta}, \\ (ii)\ \forall\varepsilon \in (0,1], \forall K \Subset \mathbb{R}^2, \exists\nu_K > 0, \exists a_\varepsilon > 0, a_{K,\varepsilon} \leq \nu_K a_\varepsilon, \\ (iii)\ \forall\varepsilon \in (0,1], \forall K \Subset \mathbb{R}^2, \exists\xi_{K,l} \geq 0, \exists P_\varepsilon > 0, \\ \sup_{x\in K_2,\ k\leq l} \left| \left(l_\varepsilon^{-1}\right)^{(k)}(x) \right| = p_{K_2,l}\left(l_\varepsilon^{-1}\right) \leq \xi_{K,l} P_\varepsilon^l, \\ (iv)\ \forall\varepsilon \in (0,1], \forall K \Subset \mathbb{R}^2, \exists\omega_K > 0, \exists Q_\varepsilon > 0, Q_{K,\varepsilon} \leq \omega_K Q_\varepsilon \\ (v)\ \mathcal{C} = A/I_A \text{ is overgenerated by the families} \\ \left(M_{\varepsilon,\eta}\right)_{\varepsilon,\eta,\rho}, \left(\rho^{Q_\varepsilon}\right)_{\varepsilon,\eta,\rho}, \left(P_\varepsilon\right)_{\varepsilon,\eta,\rho}, (\exp M_{\varepsilon,\eta} a_\varepsilon)_{\varepsilon,\eta,\rho}. \end{cases}$$

where F_η which regularize the nonlipschitzian F is given by (5), l_ε which regularize the characteristic situation is specified in $\mathbf{H_{car}}$ **hypothesis**, f_ρ which regularize the data and ρ^{Q_ε} are specified in (8); K_ε, $K_{1\varepsilon}$, K_2, a_ε and $M_{\varepsilon,\eta}$ are defined in Subsection 4.2.

The following result give a positive answer to the question of the consistency of the hypotheses (H) by giving the existence of the families overgenerating this ring in a paradigmatic example.

Proposition 10 *Assume that* $r_\eta = \dfrac{1}{\eta}$, $l_\varepsilon(t) = \varepsilon t$, $\operatorname{supp} T$ *is compact and* $F(t,x,z) = z^2$, *then*

a) (i) is fulfilled with $M_{\varepsilon,\eta} = \dfrac{1}{\eta^2}$.

b) (ii) and (iii) are fulfilled with $a_\varepsilon = P_\varepsilon = \dfrac{1}{\varepsilon}$.

c)(iv) is fulfilled with $Q_\varepsilon = Q$ *(independent of* ε*) for* ε *small enough.*

d) (v) $\mathcal{C} = A/I_A$ *is overgenerated only by the families* $(\rho)_{\varepsilon,\eta,\rho}$ *and* $(\exp \dfrac{1}{\varepsilon\eta^2})_{\varepsilon,\eta,\rho}$.

Proof. According to Corollary 7 there exists a sequence μ_n such that $\left|\frac{d^n F_\eta}{dz^n}(z)\right| \le \mu_n \frac{1}{\eta^2}$. Then a) is proved It is easy to compute $a_{K,\varepsilon} = 2b\frac{1}{\varepsilon}$, moreover, we have here: $K_{1\varepsilon} = [-b/\varepsilon, b/\varepsilon]$, $K_2 = [-b, b]$, $l_\varepsilon(t) = \varepsilon t$ and $l_\varepsilon^{-1}(x) = x/\varepsilon$. Then, b) is immediately proved. As supp$T$ is compact, the supports of all regularized f_ρ are contained in a fixed compact itself contained in $K_{1\varepsilon}$ for ε small enough: According to (8), this implies that $\rho^{Q_\varepsilon} = \rho^Q$ for ε small enough where Q is independent of ε, and c) is proved. Finally the trivial estimates

$$\rho^Q \le \rho, \frac{1}{\varepsilon} \le \exp\frac{1}{\varepsilon\eta^2}, \frac{1}{\eta^2} \le \exp\frac{1}{\varepsilon\eta^2}$$

prove that the families $(\rho)_{\varepsilon,\eta,\rho}$ and $(\exp\frac{1}{\varepsilon\eta^2})_{\varepsilon,\eta,\rho}$ are sufficient to overgenerate $\mathcal{C}$. ■

4.3.2 Existence of a solution to (P_g)

Classically, in the Hadamard sense, a Cauchy problem is well posed if existence and uniqueness of the solution to the problem is secured and, in addition, if this solution depends continuously from the data. We shall discuss below how this concept can be adapted in the setting of generalized functions for problems admitting various types of singularities.

The first step is to obtain existence of a solution to the generalized problem associated to the ill posed Cauchy one.

Theorem 11 *Under Assumption (H), the problem (P_g) admits $[u_{\varepsilon,\eta,\rho}]_{\mathcal{A}(\mathbb{R}^2)}$ as solution where $u_{\varepsilon,\eta,\rho}$ is the solution given in Proposition 9.*

Proof. Take K a compact subset of $\mathbb{R}^2$. Consider the compact subset K_ε built as in Section 4.2. From Proposition 9, we have

$$\|u_{\varepsilon,\eta,\rho}\|_{\infty,K} \le a_\varepsilon \|F_\eta(\cdot,\cdot,0)\|_{\infty,K_\varepsilon} \exp(m_{\varepsilon,\eta}a_\varepsilon) + \|f_\rho\|_{\infty,K_{1\varepsilon}} \exp(m_{\varepsilon,\eta}a_\varepsilon)$$

Thus

$$P_{K,0}(u_{\varepsilon,\eta,\rho}) \le c_{1\varepsilon\eta} + c_{2\varepsilon\eta} P_{K_{1\varepsilon},0}(f_\rho) \tag{10}$$

with $c_{1\varepsilon\eta} = a_\varepsilon \|F_\eta(\cdot,\cdot,0)\|_{\infty,K_\varepsilon} \exp(m_{\varepsilon,\eta}a_\varepsilon)$ and $c_{2\varepsilon\eta} = \exp(m_{\varepsilon,\eta}a_\varepsilon)$. On the other hand

$$F_\eta(t,x,u_{\varepsilon,\eta,\rho}(t,x)) = F_\eta(t,x,0) + u_{\varepsilon,\eta,\rho}(t,x)\int_0^1 \frac{\partial F_\eta}{\partial z}(t,x,\sigma u_{\varepsilon,\eta,\rho}(t,x)\,\mathrm{d}\sigma.$$

We have

$$|F_\eta(t,x,u_{\varepsilon,\eta,\rho}(t,x))| \le |F_\eta(t,x,0)| + m_{\varepsilon,\eta}\|u_{\varepsilon,\eta,\rho}\|_{\infty,K_\varepsilon} \le \|F(\cdot,\cdot,0)\|_{\infty,K_\varepsilon} + m_{\varepsilon,\eta}\|u_{\varepsilon,\eta,\rho}\|_{\infty,K_\varepsilon}.$$

Then, according to (10), we have

$$P_{K_\varepsilon,0}(F_\eta(\cdot,\cdot,u_{\varepsilon,\eta,\rho})) \le c_{3\varepsilon\eta} + c_{4\varepsilon\eta}P_{K_{1\varepsilon},0}(f_\rho), \tag{11}$$

with $c_{3\varepsilon\eta} = \|F(\cdot,\cdot,0)\|_{\infty,K_\varepsilon} + m_{\varepsilon,\eta}c_{1\varepsilon\eta}$ and $c_{4\varepsilon\eta} = m_{\varepsilon,\eta}c_{2\varepsilon\eta}$. Observe that $c_{i\varepsilon\eta} \in |A|$ for $i = 1,..4$. Recalling that

$$\frac{\partial u_{\varepsilon,\eta,\rho}}{\partial t}(t,x) = F_\eta(t,x,u_{\varepsilon,\eta,\rho}(t,x)),$$

we have

$$P_{K_\varepsilon,(1,0)}(u_{\varepsilon,\eta,\rho}) \le P_{K_\varepsilon,0}(F_\eta(\cdot,\cdot,u_{\varepsilon,\eta,\rho})) \le c_{3\varepsilon\eta} + c_{4\varepsilon\eta}P_{K_{1\varepsilon},0}(f_\rho).$$

We look for the estimate concerning $P_{K_\varepsilon,(0,1)}(u_{\varepsilon,\eta,\rho})$. From the integral expression (9) of $u_{\varepsilon,\eta,\rho}$ we deduce

$$\begin{aligned}\frac{\partial u_{\varepsilon,\eta,\rho}}{\partial x}(t,x) &= \left(\left(l_\varepsilon^{-1}\right)'(x)\right) f'_\rho(l_\varepsilon^{-1}(x)) + \int_0^t \frac{\partial}{\partial x}(F_\eta(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x)))\,\mathrm{d}\tau \\ &\quad - \left(\left(l_\varepsilon^{-1}\right)'(x)\right) F_\eta(l_\varepsilon^{-1}(x),x,u_{\varepsilon,\eta,\rho}(l_\varepsilon^{-1}(x),x)),\end{aligned}$$

then

$$\begin{aligned}\frac{\partial u_{\varepsilon,\eta,\rho}}{\partial x}(t,x) &= \int_0^t \left(\frac{\partial F_\eta}{\partial x}(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x)) + \frac{\partial F_\eta}{\partial z}(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x))\frac{\partial u_{\varepsilon,\eta,\rho}}{\partial x}(\tau,x)\right)\mathrm{d}\tau \\ &\quad - \left(\left(l_\varepsilon^{-1}\right)'(x)\right) F_\eta(l_\varepsilon^{-1}(x),x,u_{\varepsilon,\eta,\rho}(l_\varepsilon^{-1}(x),x)) + \left(\left(l_\varepsilon^{-1}\right)'(x)\right) f'_\rho(l_\varepsilon^{-1}(x)).\end{aligned}$$

As

$$\sup_{(t,x)\in K_\varepsilon}\left|\frac{\partial F_\eta}{\partial z}(t,x,u_{\varepsilon,\eta,\rho}(t,x))\right| = m_{\varepsilon,\eta} \leq M_{K_\varepsilon,\eta,1},$$

thus

$$\begin{aligned}\left|\frac{\partial u_{\varepsilon,\eta,\rho}}{\partial x}(t,x)\right| &\leq \int_0^t M_{K_\varepsilon,\eta,1}\,\mathrm{d}\tau + \int_0^t M_{K_\varepsilon,\eta,1}\left|\frac{\partial u_{\varepsilon,\eta,\rho}}{\partial x}(\tau,x)\right|\mathrm{d}\tau \\ &\quad + \left|\left(l_\varepsilon^{-1}\right)'(x)\right|\left(c_{3\varepsilon\eta} + c_{4\varepsilon\eta}P_{K_{1\varepsilon},0}(f_\rho)\right) + \left|\left(l_\varepsilon^{-1}\right)'(x)\right| \sup\nolimits_{K_{1\varepsilon}}\left|f'_\rho(x)\right|.\end{aligned}$$

As $\sup_{K_{2\varepsilon}}\left|\left(l_\varepsilon^{-1}\right)'(x)\right| \leq \xi_1 P_\varepsilon$, then

$$\left|\frac{\partial u_{\varepsilon,\eta,\rho}}{\partial x}(t,x)\right| \leq M_{K_\varepsilon,\eta,1}\int_0^t \left|\frac{\partial u_{\varepsilon,\eta,\rho}}{\partial x}(\tau,x)\right|\mathrm{d}\tau + p(f_\rho,K_\varepsilon),$$

with

$$p(f_\rho,K_\varepsilon) = a_\varepsilon M_{K_\varepsilon,\eta,1} + \xi_1 P_\varepsilon\left(c_{3\varepsilon\eta} + c_{4\varepsilon\eta}p_{K_{1\varepsilon},0}(f_\rho)\right) + \xi_1 P_\varepsilon p_{K_{1\varepsilon},1}(f_\rho) \in |A|.$$

To estimate the expression $\left|\frac{\partial u_{\varepsilon,\eta,\rho}}{\partial x}(t,x)\right|$ which lies on the right side of the previous inequality in an integral form, we cannot use a similar technique to the first case but have to involve the Gronwall lemma which gives

$$\begin{aligned}\left|\frac{\partial u_{\varepsilon,\eta,\rho}}{\partial x}(t,x)\right| &\leq p(f_\rho,K_\varepsilon)\exp\left(\int_0^t M_{K_\varepsilon,\eta,1}\,\mathrm{d}\tau\right) \\ &\leq p(f_\rho,K_\varepsilon)\exp\left(a_\varepsilon M_{K_\varepsilon,\eta,1}\right).\end{aligned}$$

Then

$$P_{K_\varepsilon,(0,1)}(u_{\varepsilon,\eta,\rho}) \leq p(f_\rho,K_\varepsilon)\exp\left(a_\varepsilon M_{K_\varepsilon,\eta,1}\right).$$

Now we have to compute the estimates corresponding to the cross derivatives, for all n and m,

$$\frac{\partial^{n+m+1}u_{\varepsilon,\eta,\rho}}{\partial t^{n+1}\partial^m x}(t,x) = \frac{\partial^{n+m}}{\partial t^n\partial x^m}F_\eta(t,x,u_{\varepsilon,\eta,\rho}(t,x)).$$

We have

$$\begin{aligned}(12)\qquad &\frac{\partial^{n+m}}{\partial t^n\partial x^m}F_\eta(t,x,u_{\varepsilon,\eta,\rho}(t,x)) \\ &= \sum_{|\beta|\leq n+m} c_\beta\left(D^\beta F_\eta\right)(t,x,u_{\varepsilon,\eta,\rho}(t,x))\sum_{i=1}^{n+m}\sum_{p_i(\alpha,\beta)} d_{i,\alpha,\beta}\prod_{j=1}^{i}\left(D^{l_j}u_{\varepsilon,\eta,\rho}\right)^{k_j}(t,x) + \\ &\quad \frac{\partial F_\eta}{\partial z}(t,x,u_{\varepsilon,\eta,\rho}(t,x)\left(D^{(n,m)}u_{\varepsilon,\eta,\rho}\right)(t,x).\end{aligned}$$

Thus, according to Lemma 5, for each (n,m) such that $n+m=l$, we have

$$\left|\frac{\partial^{n+m}}{\partial t^n \partial x^m} F_\eta(t,x,u_{\varepsilon,\eta,\rho}(t,x))\right| \leq C_{(n,m)} \sum_{p(\alpha)} c_\beta M_{K_\varepsilon,\eta,|\beta|} \left(P_{K,n+m-1}(u_{\varepsilon,\eta,\rho})\right)^{|\beta|} + m_{\varepsilon,\eta} \left|\frac{\partial^{n+m} u_{\varepsilon,\eta,\rho}}{\partial t^n \partial^m x}(t,x)\right|,$$

then

$$\left|\frac{\partial^{n+m}}{\partial t^n \partial x^m} F_\eta(t,x,u_{\varepsilon,\eta,\rho}(t,x))\right| \leq \Psi(P_{K_\varepsilon,n+m-1}(u_{\varepsilon,\eta,\rho})) + m_{\varepsilon,\eta} \left|\frac{\partial^{n+m} u_{\varepsilon,\eta,\rho}}{\partial t^n \partial^m x}(t,x)\right| \tag{13}$$

where $\Psi(P_{K_\varepsilon,n+m-1}(u_{\varepsilon,\eta,\rho}))$ is a polynomial in $P_{K_\varepsilon,n+m-1}(u_{\varepsilon,\eta,\rho})$ and with coefficients (expressed with terms like $c_\beta M_{K_\varepsilon,\eta,|\beta|}$) the family of which lies in $|A|$. We deduce that, for each (n,m) such that $n+m=l$, we have

$$P_{K_\epsilon,(n,m)}(F_\eta(\cdot,\cdot,u_{\varepsilon,\eta,\rho}))) \leq \Psi(P_{K_\varepsilon,n+m-1}(u_{\varepsilon,\eta,\rho})) + m_{\varepsilon,\eta} P_{K_\varepsilon,n+m}(u_{\varepsilon,\eta,\rho}). \tag{14}$$

As

$$P_{K_\varepsilon,(n+1,m)}(u_{\varepsilon,\eta,\rho}) = P_{K_\varepsilon,(n,m)}(F_\eta(\cdot,\cdot,u_{\varepsilon,\eta,\rho})),$$

then

$$P_{K_\varepsilon,(n+1,m)}(u_{\varepsilon,\eta,\rho}) \leq \Psi(P_{K_\varepsilon,n+m-1}(u_{\varepsilon,\eta,\rho})) + m_{\varepsilon,\eta} P_{K_\varepsilon,n+m}(u_{\varepsilon,\eta,\rho}). \tag{15}$$

We now proceed by induction. Suppose that, for each (p,q) such that $p+q \leq l$, we have

$$P_{K_\varepsilon,(p,q)}(u_{\varepsilon,\eta,\rho}) \leq c_{\varepsilon,l,0} + \sum_{j=1}^{p+q} c_{\varepsilon,l,j} \left(p_{K_{1\varepsilon},j}(f_\rho)\right)^{\alpha_j} \tag{16}$$

with $(c_{\varepsilon,l,j})_{\varepsilon,\eta,\rho} \in |A|$. From (15), we deduce easily that $p_{K_\varepsilon,(n+1,m)}(u_{\varepsilon,\eta,\rho})$ and $p_{K_\varepsilon,(n,m+1)}(u_{\varepsilon,\eta,\rho})$ have a similar form as (16), for each (n,m) such that $n+m=l$, $n \neq 0$. We have a finite number of couple (n,m) such that $n+m \leq l$, so we can choose the coefficients in $|A|$ such that

$$\max_{n+m=l+1} p_{K_\varepsilon,(n,m)}(u_{\varepsilon,\eta,\rho}) \leq c_{\varepsilon,l+1,0} + \sum_{j=1}^{l+1} c_{\varepsilon,l+1,j} \left(p_{K_{1\varepsilon},j}(f_\rho)\right)^{\alpha_j}. \tag{17}$$

As previously, (14), for each (n,m) such that $n+m=l$, we have

$$p_{K_\varepsilon,(n,m)}(F_\eta(\cdot,\cdot,u_{\varepsilon,\eta,\rho}))) \leq \Psi(p_{K_\varepsilon,n+m-1}(u_{\varepsilon,\eta,\rho})) + m_{\varepsilon,\eta} \sup_{(t,x)\in K_\varepsilon} \left|\frac{\partial^{n+m} u_{\varepsilon,\eta,\rho}}{\partial t^n \partial x^m}(t,x)\right|,$$

and as (17),

$$p_{K_\varepsilon,l+1}(u_{\varepsilon,\eta,\rho}) = \max_{n+m=l+1} p_{K_\varepsilon,(n,m)}(u_{\varepsilon,\eta,\rho}) \leq c_{\varepsilon,l+1,0} + \sum_{j=1}^{l+1} c_{\varepsilon,l+1,j} \left(p_{K_{1\varepsilon},j}(f_\rho)\right)^{\alpha_j}.$$

This gives the estimates for the cross derivatives.

For $m=0$ we obtain the higher order derivatives with respect to t, then, for any integer $n>0$,

$$p_{K_\varepsilon,(n,0)}(u_{\varepsilon,\eta,\rho}) \leq c_{\varepsilon,n,0} + \sum_{j=1}^{n} c_{\varepsilon,n,j} \left(p_{K_{1\varepsilon},j}(f_\rho)\right)^{\alpha_j}.$$

This leads to the first conclusion that is

$$\left(p_{K,l}(u_\lambda)\right)_\lambda \in |A|, \text{ for } n+m \leq l, n>0, m \geq 0.$$

To study the case $n=0$ we have to compute the m^{th} derivatives of $u_{\varepsilon,\eta,\rho}$ with respect to x. From the integral expression of $u_{\varepsilon,\eta,\rho}$ we deduce

$$\begin{aligned}\frac{\partial^m u_{\varepsilon,\eta,\rho}}{\partial x^m}(t,x) &= \frac{\partial^m}{\partial x^m}\left(f_\rho(l_\varepsilon^{-1}(x))\right) + \int_0^t \frac{\partial^m}{\partial x^m}F_\eta(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau \\ &\quad - \sum\nolimits_{j=0}^{m-1} C_{m-1}^j \left(\left(l_\varepsilon^{-1}\right)^{(m-j+1)}(x)\right)\frac{\partial^j}{\partial x^j}F_\eta(l_\varepsilon^{-1}(x),x,u_{\varepsilon,\eta,\rho}(l_\varepsilon^{-1}(x),x)).\end{aligned} \tag{18}$$

According to (13), we have

$$\left|\frac{\partial^m}{\partial x^m}F_\eta(t,x,u_{\varepsilon,\eta,\rho}(t,x))\right| \le \Psi(p_{K_\varepsilon,m-1}(u_{\varepsilon,\eta,\rho})) + m_{\varepsilon,\eta}\left|\frac{\partial^m u_{\varepsilon,\eta,\rho}}{\partial x^m}(t,x)\right|. \tag{19}$$

We now proceed by induction. Suppose that

$$p_{K_\varepsilon,(0,l)}(u_{\varepsilon,\eta,\rho}) \le c_{\varepsilon,l,0} + \sum_{j=1}^{l} c_{\varepsilon,l,j}\left(p_{K_{1\varepsilon},j}(f_\rho)\right)^{\alpha_j} \tag{20}$$

for any $(0,l)$ such that $l \le m-1$. Then from (19) we can deduce that exists $\left(c'_{\varepsilon,l,j}\right)_{\varepsilon,\eta,\rho} \in |A|$ such that

$$\left|\frac{\partial^m}{\partial x^m}F_\eta(t,x,u_{\varepsilon,\eta,\rho}(t,x))\right| \le \sum_{j=0}^{m-1} c'_{\varepsilon,m-1,j}\left(p_{K_{1\varepsilon},,j}(f_\rho)\right)^{\alpha_j} + m_{\varepsilon,\eta}\left|\frac{\partial^m u_{\varepsilon,\eta,\rho}}{\partial x^m}(t,x)\right|.$$

We set

$$\sum_{j=0}^{m-1} c'_{\varepsilon,m-1,j}\left(p_{K_{1\varepsilon},j}(f_\rho)\right)^{\alpha_j} = q_{1,m}(f_\rho).$$

According to Faà di Bruno's formula, we have

$$\frac{\partial^m}{\partial x^m}\left(f_\rho(l_\varepsilon^{-1}(x))\right) = \sum_{(m_1,\ldots,m_n)} t^n_{m_1,\ldots,m_n}\frac{\partial^{(m_1+\ldots+m_n)}f_\rho}{\partial z^{(m_1+\ldots+m_n)}}(l_\varepsilon^{-1}(x))\prod_{j:m_j\neq 0}\left(l_\varepsilon^{-1}\right)^{(m_j)}(x)$$

where the coefficients $t_{i_1,\ldots,i_m}$ are integers and the n-uples $(m_1,\ldots,m_n)$ satisfy the constraint $\sum_{k=0}^n km_k = m$. From the hypothesis (H), part (iii), we deduce, for $m_j\in\mathbb{N}$, that

$$\sup_{K_{2\varepsilon}}\left|\left(l_\varepsilon^{-1}\right)^{(m_j)}(x)\right| \le \xi_{m_j}P_\varepsilon^{m_j}.$$

Thus, for $x\in K_{2\varepsilon}$, we have

$$\begin{aligned}\left|\frac{\partial^m}{\partial x^m}\left(f_\rho(l_\varepsilon^{-1}(x))\right)\right| &\le \sum_{(m_1,\ldots,m_n)} t^n_{m_1,\ldots,m_n}\left|\frac{\partial^{(m_1+\ldots+m_n)}f_\rho}{\partial z^{(m_1+\ldots+m_n)}}(l_\varepsilon^{-1}(x))\right|\prod_{j:m_j\neq 0}\xi_{m_j}P_\varepsilon^{m_j} \\ &\le \sum_{(m_1,\ldots,m_n)} t^n_{m_1,\ldots,m_n}p_{K_{1\varepsilon},m_1+\ldots+m_n}(f_\rho)\prod_{j:m_j\neq 0}\xi_{m_j}P_\varepsilon^{m_j} \\ &\le \sum_{(m_1,\ldots,m_n)} t^n_{m_1,\ldots,m_n}p_{K_{1\varepsilon},m}(f_\rho)\prod_{j:m_j\neq 0}\xi_{m_j}P_\varepsilon^{m_j}\end{aligned}$$

which is a polynomial $\chi_2\left(p_{K_{1\varepsilon},m}(f_\rho)\right) = q_{3,m}(f_\rho)$.

We Set

$$\ominus_\eta(t,x) = F_\eta(t,x,u_{\varepsilon,\eta,\rho}(t,x)).$$

Then

$$\frac{\partial^j}{\partial x^j}F_\eta(l_\varepsilon^{-1}(x),x,u_{\varepsilon,\eta,\rho}(l_\varepsilon^{-1}(x),x))=\frac{\partial^j}{\partial x^j}\ominus_\eta(l_\varepsilon^{-1}(x),x)$$

but, according to ([26]),

$$\frac{\partial^j}{\partial x^j}\ominus_\eta(l_\varepsilon^{-1}(x),x)=\sum_{k=1}^{j}t_{j,k}(l_\varepsilon^{-1})^{(k)}(x)\frac{\partial}{\partial t}\frac{\partial^{j-k}\ominus_\eta}{\partial x^{j-k}}(l_\varepsilon^{-1}(x),x)+\frac{\partial^j\ominus_\eta}{\partial x^j}(l_\varepsilon^{-1}(x),x)$$

where $t_{j,k}=\frac{(j-1)!}{(j-k)!(k-1)!}$. Moreover, for $0\leq j\leq m-1$, we have

$$\sup_{K_{1\varepsilon}}\left|\frac{\partial}{\partial t}\frac{\partial^{j-k}\ominus_\eta}{\partial x^{j-k}}(l_\varepsilon^{-1}(x),x)\right|\leq\sup_{(t,x)\in K_\varepsilon}\left|\frac{\partial}{\partial t}\frac{\partial^{j-k}}{\partial x^{j-k}}F_\eta(t,x,u_{\varepsilon,\eta,\rho}(t,x))\right|$$

is bounded by a polynomial $p\left(p_{K_{1\varepsilon},j-k+1}(f_\rho)\right)=p_{j-k+1}\left(K_{1\varepsilon},f_\rho\right)$ and

$$\sup_{K_{1\varepsilon}}\left|\frac{\partial^j\ominus_\eta}{\partial x^j}(l_\varepsilon^{-1}(x),x)\right|\leq\sup_{(t,x)\in K_\varepsilon}\left|\frac{\partial^j}{\partial x^j}F_\eta(t,x,u_{\varepsilon,\eta,\rho}(t,x))\right|$$

is bounded by a polynomial $p\left(p_{K_{1\varepsilon},j}(f_\rho)\right)=p_j\left(K_{1\varepsilon},f_\rho\right)$. We deduce that

$$\left|\frac{\partial^j}{\partial x^j}\ominus_\eta(l_\varepsilon^{-1}(x),x)\right|\leq\sum_{k=1}^{j}t_{j,k}\xi_{k_j}P_\varepsilon^{k_j}p_{j-k+1}\left(K_{1\varepsilon},f_\rho\right)+p_j\left(K_{1\varepsilon},f_\rho\right).$$

Then

$$\begin{aligned}&\left|\sum\nolimits_{j=0}^{m-1}C_{n-1}^{j}\left(\left(l_\varepsilon^{-1}\right)^{(m-j+1)}(x)\right)\frac{\partial^j}{\partial x^j}\ominus_\eta(l_\varepsilon^{-1}(x),x)\right|\\&\leq\sum\nolimits_{j=0}^{m-1}C_{n-1}^{j}\xi_{m-j+1}P_\varepsilon^{m-j+1}\left|\frac{\partial^j}{\partial x^j}\ominus_\eta(l_\varepsilon^{-1}(x),x)\right|\end{aligned}$$

is bounded by a polynomial $p\left(p_{K_{1\varepsilon},m-1}(f_\rho)\right)=q_{2,m}(f_\rho)$.

We deduce from (18) the following estimate

$$\begin{aligned}\left|\frac{\partial^m u_{\varepsilon,\eta,\rho}}{\partial x^m}(t,x)\right|\leq&\left|\frac{\partial^m}{\partial x^m}\left(f_\rho(l_\varepsilon^{-1}(x))\right)\right|+\int_0^t\left(q_{1,m}(f_\rho)+m_{\varepsilon,\eta}\left|\frac{\partial^m u_{\varepsilon,\eta,\rho}}{\partial x^m}(t,x)\right|\right)\mathrm{d}\tau,\\&+q_{2,m}(f_\rho),\end{aligned}$$

then

$$\begin{aligned}\left|\frac{\partial^m u_{\varepsilon,\eta,\rho}}{\partial x^m}(t,x)\right|&\leq q_{3,m}(f_\rho)+\int_0^t q_{1,m}(f_\rho)\,\mathrm{d}\tau+m_{\varepsilon,\eta}\int_0^t\left|\frac{\partial^m u_{\varepsilon,\eta,\rho}}{\partial x^m}(t,x)\right|\mathrm{d}\tau+q_{2,m}(f_\rho)\\&\leq m_{\varepsilon,\eta}\int_0^t\left|\frac{\partial^m u_{\varepsilon,\eta,\rho}}{\partial x^m}(t,x)\right|\mathrm{d}\tau+q_m(f_\rho),\end{aligned}$$

where

$$q_m(f_\rho)=q_{3,m}(f_\rho)+a_\varepsilon q_{1,m}(f_\rho)+q_{2,m}(f_\rho).$$

So we have again, from the Gronwall lemma,

$$\left|\frac{\partial^m u_{\varepsilon,\eta,\rho}}{\partial x^m}(t,x)\right|\leq q_m(f_\rho)\exp\left(\int_0^t m_{\varepsilon,\eta}\,\mathrm{d}\tau\right).\tag{21}$$

Thus

$$\left|\frac{\partial^m u_{\varepsilon,\eta,\rho}}{\partial x^m}(t,x)\right| \leq q_m(f_\rho)\exp\left(a_\varepsilon m_{\varepsilon,\eta}\right).$$

Then we have

$$p_{K_\varepsilon,(0,m)}(u_{\varepsilon,\eta,\rho}) \leq q_m(f_\rho)\exp\left(a_\varepsilon m_{\varepsilon,\eta}\right).$$

Thus $p_{K_\varepsilon,(0,m)}(u_{\varepsilon,\eta,\rho})$ is bounded by a polynomial $\Psi((p_{K_{1\varepsilon},m}(f_\rho)))$ with coefficients the family of which lies in $|A|$.

Finally, for each (n,m) such that $n+m \leq l$, we can maintain that $p_{K_\varepsilon,(n,m)}(u_{\varepsilon,\eta,\rho})$ is bounded by such a polynomial and $(p_{K,l}(u_\lambda))_\lambda \in |A|$. Then $[u_\lambda] \in \mathcal{A}\left(\mathbb{R}^2\right)$. $\blacksquare$

4.3.3 Uniqueness of the solution to (P_g)

Theorem 12 *Under Assumption (H):*

(i) The problem (P_g) admits $[u_{\varepsilon,\eta,\rho}]_{\mathcal{A}(\mathbb{R}^2)}$ as solution where $u_{\varepsilon,\eta,\rho}$ is given in Proposition 9.;

(ii) Without further assumption this solution is (in general) not unique.

Proof. The proof of the assertion(i) is giving in Theorem 11. The assertion (ii) comes from a counter example given by M. Oberguggenberger for the purely characteristic case [8]. In this case, studied in a one parametrized algebra, uniqueness is recovered by working in the algebra $\mathcal{G}_\tau\left(\mathbb{R}^2\right)$ of tempered generalized functions.(Definitions 3, 4). $\blacksquare$

As mentioned above, we have to add some hypothesis and additional constructions to obtain uniqueness. The natural topology of $\mathcal{O}_M$ permits a new approach of algebras of tempered generalized function, $\mathcal{G}_{\mathcal{O}_M}$ [7] which differs to $\mathcal{G}_\tau$ [15] but permits a point value characterization (H. Vernaeve, personal communication, 2009, [30]) and an extension $\mathcal{A}_{O_M}$ in the framework of $(\mathcal{C}$,$\mathcal{E}$,$\mathcal{P})$-algebras [17]. Our goal is to recover uniqueness of the solution of (P_g) in the context of $(\mathcal{C}$,$\mathcal{E}$,$\mathcal{P})$- type extensions of $\mathcal{O}_M$.

Remark 6 *Note that for linear (or semi linear) problems with irregular data, a more complete theory exists, based on the functorial properties of the Colombeau type algebras [10]. Existence and uniqueness are obtained whenever the map associating the solution to the data for the classical problem is continuously temperate. Of course, this theory fails when the problem under consideration is characteristic. (See [8] for a complete discussion on this subject.)*

Proposition 13 *The solution of (P_g) is unique in $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$.*

Proof. We recover uniqueness of the solution of (P_g) by working in algebras $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^d\right)$ of algebras of tempered generalized functions

Take $S(\cdot)$ the presheaf of rapidly decreasing smooth functions.

$$S(\mathbb{R}^n) = \left\{f \in \mathrm{C}^\infty(\mathbb{R}^n) : \forall (q,l) \in \mathbb{N}^2, \mu_{q,l}(f) < +\infty\right\}$$

For $\varphi \in S(\mathbb{R}^n)$, $f \in \mathrm{C}^\infty(\mathbb{R}^n)$ and $l \in \mathbb{N}$, we put

$$\upsilon_{\varphi,l}(f) = \sup_{x\in\mathbb{R}^n, |\alpha|\leq l} |\varphi(x)\mathcal{D}^\alpha f(x)|$$

and

$$\mathcal{X}_{\mathcal{O}_M}\left(\mathbb{R}^n\right) = \{(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^n)^{(0,1]} : \forall \varphi \in S(\mathbb{R}^n), \forall l \in \mathbb{N}, \exists m \in \mathbb{N}, \upsilon_{\varphi,l}(f_\varepsilon) = O(\varepsilon^{-m}) \quad (\varepsilon \to 0)\},$$
$$\mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^n\right) = \{(f_\varepsilon)_\varepsilon \in \mathcal{O}_M(\mathbb{R}^n)^{(0,1]} : \forall \varphi \in S(\mathbb{R}^n), \forall l \in \mathbb{N}, \forall m \in \mathbb{N}, \upsilon_{\varphi,l}(f_\varepsilon) = O(\varepsilon^{p}) \quad (\varepsilon \to 0)\}.$$

We have $\mathcal{X}_{\mathcal{O}_M}\left(\mathbb{R}^n\right) = \mathcal{X}_\tau\left(\mathbb{R}^n\right)$ (see [7]). Take $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^n\right) = \mathcal{X}_{\mathcal{O}_M}\left(\mathbb{R}^n\right) / \mathcal{N}_{\mathcal{O}_M}\left(\mathbb{R}^n\right)$.

Indeed a straightforward calculation shows that $(u_{\varepsilon,\eta,\rho})_{\varepsilon,\eta,\rho} \in \mathcal{X}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$. Thus the class of $(u_{\varepsilon,\eta,\rho})_{\varepsilon,\eta,\rho}$ in $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$ is a solution to problem (P_g) in $\mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$.

Let $v = [v_{\varepsilon,\eta,\rho}] \in \mathcal{G}_{\mathcal{O}_M}\left(\mathbb{R}^2\right)$ be another solution to (P_g). According to (P_{rep}) we have

$$\left\{ \begin{array}{c} \dfrac{\partial}{\partial t}(v_{\varepsilon,\eta,\rho}(t,x)) = F_\eta(t,x,v_{\varepsilon,\eta,\rho}(t,x)) \\ v_{\varepsilon,\eta,\rho}(t,l_\varepsilon(t)) = f_\rho(t) + j_{\varepsilon,\eta,\rho}(t) \end{array} \right.$$

with $(j_{\varepsilon,\eta,\rho})_{\varepsilon,\eta,\rho} \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$. Then

$$v_{\varepsilon,\eta,\rho}(t,x) = f_\rho(l_\varepsilon^{-1}(x)) + j_{\varepsilon,\eta,\rho}(l_\varepsilon^{-1}(x)) + \int_{l_\varepsilon^{-1}(x)}^t F_\eta(\tau,x,v_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau.$$

When putting $w_{\varepsilon,\eta,\rho} = (v_{\varepsilon,\eta,\rho} - u_{\varepsilon,\eta,\rho})$ we get

$$w_{\varepsilon,\eta,\rho}(t,x) = j_{\varepsilon,\eta,\rho}(l_\varepsilon^{-1}(x)) + \int_{l_\varepsilon^{-1}(x)}^t (F_\eta(\tau,x,v_{\varepsilon,\eta,\rho}(\tau,x)) - F_\eta(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x)))\,\mathrm{d}\tau$$

but

$$\begin{aligned} &F_\eta(\tau,x,v_{\varepsilon,\eta,\rho}(\tau,x)) - F_\eta(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x)) \\ &\qquad = w_{\varepsilon,\eta,\rho}(\tau,x)\left(\int_0^1 \frac{\partial F_\eta}{\partial z}(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x) + \theta w_{\varepsilon,\eta,\rho}((\tau,x))\,\mathrm{d}\theta\right), \end{aligned}$$

so

$$w_{\varepsilon,\eta,\rho}(t,x) = j_{\varepsilon,\eta,\rho}(l_\varepsilon^{-1}(x)) + \int_{l_\varepsilon^{-1}(x)}^t w_{\varepsilon,\eta,\rho}(\tau,x)\left(\int_0^1 \frac{\partial F_\eta}{\partial z}(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x) + \theta w_{\varepsilon,\eta,\rho}((\tau,x))\,\mathrm{d}\theta\right)\mathrm{d}\tau.$$

Take K a compact subset of $\mathbb{R}^2$. Consider the compact subset K_ε built as in Section 4.2. Let $(t,x) \in K_\varepsilon$ and $\varphi \in S(\mathbb{R}^2)$, we have

$$|\varphi(t,x)w_{\varepsilon,\eta,\rho}(t,x)| \le m_{\varepsilon,\eta}\int_{l_\varepsilon^{-1}(x)}^t |\varphi(\tau,x)w_{\varepsilon,\eta,\rho}(\tau,x)|\,\mathrm{d}\tau + Y_{\varepsilon,\eta,\rho},$$

where $Y_{\varepsilon,\eta,\rho} = \sup_{(t,x)\in K_\varepsilon}\left|\varphi(t,x)j_{\varepsilon,\eta,\rho}(l_\varepsilon^{-1}(x))\right|$. As $(j_{\varepsilon,\eta,\rho})_{\varepsilon,\eta,\rho} \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R})$ thus $Y_{\varepsilon,\eta,\rho} \in |I_A|$. Put $e(\tau) = \sup_{x\in K_{2\varepsilon}}|\varphi(\tau,x)w_{\varepsilon,\eta,\rho}(\tau,x)|$. Thus

$$|\varphi(t,x)w_{\varepsilon,\eta,\rho}(t,x)| \le m_{\varepsilon,\eta}\int_{l_\varepsilon^{-1}(x)}^t e(\tau)\,\mathrm{d}\tau + Y_{\varepsilon,\eta,\rho}.$$

We deduce that

$$\forall t \in K_{1\varepsilon},\ e(t) \le m_{\varepsilon,\eta}\int_{l_\varepsilon^{-1}(x)}^t e(\tau)\,\mathrm{d}\tau + Y_{\varepsilon,\eta,\rho}.$$

Thus, according to Gronwall's lemma,

$$\forall t \in K_{1\varepsilon},\ e(t) \le \exp\left(\int_{l_\varepsilon^{-1}(x)}^t m_{\varepsilon,\eta}\,\mathrm{d}\tau\right)Y_{\varepsilon,\eta,\rho},$$

thus

$$e(t) \le \exp(a_\varepsilon m_{\varepsilon,\eta})Y_{\varepsilon,\eta,\rho}$$

and consequently

$$\upsilon_{\varphi,0}(w_{\varepsilon,\eta,\rho}) \leq \exp(a_\varepsilon m_{\varepsilon,\eta}) Y_{\varepsilon,\eta,\rho}$$

which leads immediately to

$$\left(\upsilon_{\varphi,0}(w_{\varepsilon,\eta,\rho})\right)_{\varepsilon,\eta,\rho} \in |I_A| \,.$$

This implies the 0^{th} order estimate. Indeed the set B is stable by inverse and contains the element $(\rho)_{\varepsilon,\eta,\rho}$ such that $\lim_\Lambda \rho = 0$. It follows that $\left(\upsilon_{\varphi,l}(w_{\varepsilon,\eta,\rho})\right)_{\varepsilon,\eta,\rho} \in |I_A|$ for any $l \in \mathbb{N}$, thus $(w_{\varepsilon,\eta,\rho})_{\varepsilon,\eta,\rho} \in \mathcal{N}_{\mathcal{O}_M}(\mathbb{R}^n)$. ■

4.4 Dependence on the choice of regularization processes

Any solution to (P_g) (unique or not) depends *a priori* on the choice of regularizing processes. We expect to obtain more precise informations about this dependence. A first step in this direction is done by [1] in which the purely characteristic case is studied (with regular data). By imposing some restrictions on the asymptotical growth of the $(l_\varepsilon)_\varepsilon$ (notation of Subsubsection 4.4.2) the authors are able to prove that the generalized solution depends solely on the class of $(l_\varepsilon)_\varepsilon$ as a generalized function, not on the particular representative. They also prove that in the non-characteristic smooth case, the generalized solution provided by the method described above coincides (in the sense of generalized functions) with the classical smooth solution.

4.4.1 Independence of the generalized solution from the class of cut off functions.

Set $\Lambda_1 = (0,1]$,

$$\begin{aligned}
&\mathcal{H}_1(\mathbb{R}) = \{(f_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\mathbb{R})]^{\Lambda_1} : \forall K \Subset \mathbb{R}, \forall l \in \mathbb{N},\ (P_{K,l}(f_\varepsilon))_\varepsilon \in |A|\},\\
&\mathcal{J}_1(\mathbb{R}) = \{(f_\varepsilon)_\varepsilon \in [\mathrm{C}^\infty(\mathbb{R})]^{\Lambda_1} : \forall K \Subset \mathbb{R}, \forall l \in \mathbb{N},\ (P_{K,l}(f_\varepsilon))_\varepsilon \in |I_A|\},\\
&\mathcal{A}_1(\mathbb{R}) = \mathcal{H}_1(\mathbb{R})/\mathcal{J}_1(\mathbb{R}).
\end{aligned}$$

Consider $\mathcal{T}(\mathbb{R})$ the set of families of smooth one-variable functions $(h_\eta)_\eta \in \mathcal{H}(\mathbb{R})$, verifying the following assumptions

$$\text{(E1)} \qquad \exists\, (s_\eta)_\eta \in \mathbb{R}_*^{(0,1]} : \sup_{z\in[-s_\eta,s_\eta]} |h_\eta(z)| = 1,\ h_\eta(z) = \begin{cases} 0, \text{ if } \ |z| \geq s_\eta \\ 1, \text{ if } -s_\eta + 1 \leq z \leq s_\eta - 1 \end{cases},$$

$$\text{(E2)} \qquad \exists q \in \mathbb{N}^*, \forall\, (h_\eta)_\eta \in \mathcal{T}(\mathbb{R}), \forall \eta, s_\eta \leq r_\eta^q.$$

Moreover assume that, for every integer $n > 0$, $h_\eta^{(n)}$ is bounded on $J_\eta = [-s_\eta, s_\eta]$ independently of η.

We have $(g_\eta)_{\eta\in\Lambda_1} \in \mathcal{T}(\mathbb{R})$. Recall that $\phi_\eta(z) = z g_\eta(z)$ for $z \in \mathbb{R}$, $F_\eta(t,x,z) = F(t,x,\phi_\eta(z))$ for $(t,x,z) \in \mathbb{R}^3$ and

$$\sup_{z\in I_\eta} \left| g_\eta^{(n)}(z) \right| = M_n$$

where $I_\eta = [-r_\eta, r_\eta]$.

Let $g \in \mathcal{T}(\mathbb{R})/\mathcal{J}_1(\mathbb{R})$ be the class of $(g_\eta)_\eta$. Take $(h_\eta)_\eta$ another representative of g, that is to say $(h_\eta)_\eta \in \mathcal{T}(\mathbb{R})$, such that

$$\text{(E0)} \qquad (g_\eta - h_\eta)_\eta \in \mathcal{J}_1(\mathbb{R}).$$

Set $\sigma_\eta(z) = z h_\eta(z)$ for $z \in \mathbb{R}$, $H_\eta(t,x,z) = F(t,x,\sigma_\eta(z))$ for $(t,x,z) \in \mathbb{R}^3$ and

$$\sup_{z\in[-s_\eta,s_\eta]} \left| \frac{\partial^n h_\eta}{\partial z}(z) \right| = M'_n.$$

Our choice is made such that $(\mathrm{supp}\,(h_\eta))_\eta$ have the same growth as $(\mathrm{supp}\,(g_\eta))_\eta$ with respect to the scale $(r_\eta^q)_\eta$, in this way the corresponding solutions are lying in the same algebra $\mathcal{A}\left(\mathbb{R}^2\right)$.

Lemma 14 *Set $F \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R})$, $\phi \in \mathrm{C}^\infty(\mathbb{R}, \mathbb{R})$, $\mathfrak{F}(t,x,z) = F(t,x,\phi(z))$. For any $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\alpha_3 \geq 0$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = n \neq 0$, we have*

$$\frac{\partial^n \mathfrak{F}}{\partial t^{\alpha_1} \partial x^{\alpha_2} \partial z^{\alpha_3}}(t,x) = \sum_{1 \leq |\beta| \leq n} \left(D^\beta F\right)(t,x,\phi(z)) \sum_{i=1}^{n} \sum_{p_i(\alpha,\beta)} d_{i,\alpha,\beta} \prod_{j=1}^{i} \left(\frac{\partial^{l_j}}{\partial z^{l_j}} \phi(z)\right)^{k_j}$$

where $\beta \in \mathbb{N}^3$. The set $p_i(\alpha,\beta)$ mentioned in the inner sum consists of all nonzero multi-indices $(k_1, ..., k_i, l_1, ..., l_i) \in (\mathbb{N})^{2i}$, such that

$$0 < l_1 < ... < l_i, \quad \sum_{j=1}^{i} k_j = \beta_3, \quad \sum_{j=1}^{i} k_j l_j = \alpha_3.$$

The proof uses the Multivariate Faà di Bruno's formula (see [4]).

Corollary 15 *Set $F \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R})$, $\sigma_\eta(z) = z h_\eta(z)$ with $(h_\eta)_\eta \in \mathcal{T}(\mathbb{R})$, $H_\eta(t,x,z) = F(t,x,\sigma_\eta(z))$, $\alpha = (\alpha_1, \alpha_2, \alpha_3)$, $\alpha_1 \geq 0$, $\alpha_2 \geq 0$, $\alpha_3 \geq 0$ with $|\alpha| = \alpha_1 + \alpha_2 + \alpha_3 = n \neq 0$. Then, for $\beta \in \mathbb{N}^3$, $1 \leq |\beta| \leq n$, there exist constants $C_{|\beta|}$ which no depend of F and ϕ_η, such that $\forall K \Subset \mathbb{R}^2$, $\forall (t,x) \in K$, $\forall z \in [-s_\eta, s_\eta]$,*

$$\left|\frac{\partial^n H_\eta}{\partial t^{\alpha_1} \partial x^{\alpha_2} \partial z^{\alpha_3}}(t,x,z)\right| \leq \sum_{1 \leq |\beta| \leq n} P_{K,|\beta|}(F)\, C_{|\beta|} s_\eta^{\alpha_3}$$

Proof. We have
$$\frac{\partial^n \sigma_\eta}{\partial z^n}(z) = z \frac{\partial^n h_\eta}{\partial z^n}(z) + n \frac{\partial^{n-1} h_\eta}{\partial z^{n-1}}(z).$$
Thus $\left|\frac{\partial^n \sigma_\eta}{\partial z^n}(z)\right| \leq s_\eta M'_n + n M'_{n-1} \leq \alpha_n s_\eta \leq \alpha_n r_\eta^q$, where $\alpha_n = 2\max(M'_n; n M'_{n-1})$. So we deduce the formula. Moreover, according (E2), we have $s_\eta \leq r_\eta^q$, so

$$\left|\frac{\partial^n H_\eta}{\partial t^{\alpha_1} \partial x^{\alpha_2} \partial z^{\alpha_3}}(t,x,z)\right| \leq \sum_{1 \leq |\beta| \leq n} P_{K,|\beta|}(F)\, C_{|\beta|} r_\eta^{q\alpha_3}$$

■

Corollary 16 *Set $S_n = \left\{\alpha \in \mathbb{N}^3 : |\alpha| = n\right\}$ when $n \in \mathbb{N}^*$. Let $F \in \mathrm{C}^\infty(\mathbb{R}^3, \mathbb{R})$, H_η defined by $H_\eta(t,x,z) = F(t,x,\sigma_\eta(z))$. Assume that*

$$\forall (x,y) \in \mathbb{R}^2, F(t,x,0) = 0\ ,$$

Corollary 17
$$\exists p_0 > 0, \forall \alpha \in \mathbb{N}^3, |\alpha| = n > p_0, D^\alpha F(t,x,z) = 0,$$

(22)
$$\forall n \in \mathbb{N}, n \leq p_0, \exists d_n > 0, \forall \eta \in (0,1], \forall K \Subset \mathbb{R}^2, \sup_{(t,x) \in K;\ z \in [-r_\eta, r_\eta]; \alpha \in Sn} |D^\alpha F(t,x,z)| \leq d_n r_\eta^{p_0},$$

then $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under the family $(H_\eta)_\eta$.

Proof. Indeed, we have $\forall K \Subset \mathbb{R}^2$, $\forall (x,y) \in K$, $\forall z \in [-s_\eta, s_\eta]$, $\forall \alpha \in \mathbb{N}^3$,

$$\left|\frac{\partial^n H_\eta}{\partial t^{\alpha_1}\partial x^{\alpha_2}\partial z^{\alpha_3}}(t,x,z)\right| \leq \sum_{1\leq|\beta|\leq n} P_{K,|\beta|}(F)\, C_{|\beta|} r_\eta^{q\alpha_3} \leq \sum_{1\leq|\beta|\leq p} d_{|\beta|} r_\eta^{p_0} C_{|\beta|} r_\eta^{qp_0}$$
$$\leq \mu_n r_\eta^{p_0(1+q)}$$

where μ_n no depend to η and r_η. So, as $\sigma_\eta(z) = 0$ if $z \notin [-s_\eta, s_\eta]$,

$$\sup_{(t,x)\in K;\ z\in\mathbb{R};\alpha\in Sn} |D^\alpha H_\eta(t,x,z)| \leq \mu_n r_\eta^{p_0(1+q)},$$

and, according to the previous results, $\mathcal{A}\left(\mathbb{R}^2\right)$ is stable under the family $(H_\eta)_\eta$. ■

Theorem 18 *Assume that $p = p_0(1+q)$. Under the same hypotheses as Corollary 16, Problem (P_g), a fortiori its solution depends solely on g ($g \in \mathcal{T}(\mathbb{R})/\mathcal{J}(\mathbb{R})$) as generalized functions and not on their representatives $(g_\eta)_\eta$.*

Proof. We have associated the generalized operator $\mathcal{F}$ to F *via* the family $(g_\eta)_\eta$. Let $(h_\eta)_\eta \in (\mathrm{C}^\infty(\mathbb{R}))^{\Lambda_2}$ another family representative of the class $[g_\eta] = g$ and leading to another generalized operator $\mathcal{L}$ associated to F. We have to prove that $\mathcal{L} = \mathcal{F}$, that is to say $\mathcal{L}(u) = \mathcal{F}(u)$ for any $u \in \mathcal{A}(\mathbb{R}^2)$. Then, in terms of representatives, we have to prove that, if $(u_{\varepsilon,\eta,\rho})_{\varepsilon,\eta,\rho}$, $(v_{\varepsilon,\eta,\rho})_{\varepsilon,\eta,\rho} \in \mathcal{H}(\mathbb{R}^2)$ and $(w_{\varepsilon,\eta,\rho})_{\varepsilon,\eta,\rho} = (v_{\varepsilon,\eta,\rho} - u_{\varepsilon,\eta,\rho})_{\varepsilon,\eta,\rho} \in \mathcal{J}(\mathbb{R}^2)$, then

$$\left(F(\cdot,\cdot,\sigma_\eta(v_\lambda)) - F(\cdot,\cdot,\phi_\eta(u_{\varepsilon,\eta,\rho}))\right)_{\varepsilon,\eta,\rho} \in \mathcal{J}(\mathbb{R}^2).$$

Let

$$\Delta_{\varepsilon,\eta,\rho}(t,x) = \sigma_\eta(v_{\varepsilon,\eta,\rho}(t,x)) - \phi_\eta(u_{\varepsilon,\eta,\rho}(t,x)).$$

We have $\forall K \Subset \mathbb{R}^2$, $\forall (t,x) \in K$,

$$\Delta_{\varepsilon,\eta,\rho}(t,x) = v_{\varepsilon,\eta,\rho}(t,x) h_\eta(v_{\varepsilon,\eta,\rho}(t,x)) - u_{\varepsilon,\eta,\rho}(t,x) g_\eta(u_{\varepsilon,\eta,\rho}(t,x)),$$

so

$$\Delta_{\varepsilon,\eta,\rho}(t,x) = w_{\varepsilon,\eta,\rho}(t,x) h_\eta(v_{\varepsilon,\eta,\rho}(t,x)) + u_{\varepsilon,\eta,\rho}(t,x)\left(h_\eta(v_{\varepsilon,\eta,\rho}(t,x)) - g_\eta(u_{\varepsilon,\eta,\rho}(t,x))\right) \tag{23}$$

and

$$h_\eta \circ v_{\varepsilon,\eta,\rho} - g_\eta \circ u_{\varepsilon,\eta,\rho} = (h_\eta \circ v_{\varepsilon,\eta,\rho} - h_\eta \circ u_{\varepsilon,\eta,\rho}) + (h_\eta \circ u_{\varepsilon,\eta,\rho} - g_\eta \circ u_{\varepsilon,\eta,\rho})$$

As

$$h_\eta(v_{\varepsilon,\eta,\rho}(t,x)) - h_\eta(u_{\varepsilon,\eta,\rho}(t,x)) \tag{24}$$
$$= (v_{\varepsilon,\eta,\rho}(t,x) - u_{\varepsilon,\eta,\rho}(t,x)) \int_0^1 \frac{\partial h_\eta}{\partial z}(u_{\varepsilon,\eta,\rho}(t,x) + \mu(v_{\varepsilon,\eta,\rho}(t,x) - u_{\varepsilon,\eta,\rho}(t,x)))\,\mathrm{d}\mu, \tag{25}$$

so

$$h_\eta(v_{\varepsilon,\eta,\rho}(t,x)) - g_\eta(u_{\varepsilon,\eta,\rho}(t,x)) \tag{26}$$
$$= w_{\varepsilon,\eta,\rho}(t,x) \int_0^1 \frac{\partial h_\eta}{\partial z}(u_{\varepsilon,\eta,\rho}(t,x) + \mu w_{\varepsilon,\eta,\rho}(t,x))\,\mathrm{d}\mu + (h_\eta - g_\eta)(u_{\varepsilon,\eta,\rho}(t,x)). \tag{27}$$

We deduce that $\forall(t,x) \in K$,

$$\begin{aligned}|h_\eta(v_{\varepsilon,\eta,\rho}(t,x)) - g_\eta(u_{\varepsilon,\eta,\rho}(t,x))| &\leq |w_{\varepsilon,\eta,\rho}(t,x)| \int_0^1 M_1' \,\mathrm{d}\mu + |(h_\eta - g_\eta)(u_{\varepsilon,\eta,\rho}(t,x))| \\ &\leq |w_{\varepsilon,\eta,\rho}(t,x)| M_1' + p_{J_\eta,1}(h_\eta - g_\eta),\end{aligned}$$

where $J_\eta = [-s_\eta, s_\eta]$. Then

$$\begin{aligned}|\Delta_{\varepsilon,\eta,\rho}(t,x)| &\leq |w_{\varepsilon,\eta,\rho}(t,x)| + |u_{\varepsilon,\eta,\rho}(t,x)| \left(|w_{\varepsilon,\eta,\rho}(t,x)| M_1' + p_{J_\eta,1}(h_\eta - g_\eta)\right) \\ &\leq |w_{\varepsilon,\eta,\rho}(t,x)| \left(1 + |u_{\varepsilon,\eta,\rho}(t,x)| M_1'\right) + |u_{\varepsilon,\eta,\rho}(t,x)| p_{J_\eta,1}(h_\eta - g_\eta).\end{aligned}$$

Consequently

$$|\Delta_{\varepsilon,\eta,\rho}(t,x)| \leq P_{K,0}(w_{\varepsilon,\eta,\rho}) \left(1 + P_{K,0}(u_{\varepsilon,\eta,\rho}) M_1'\right) + P_{K,0}(u_{\varepsilon,\eta,\rho}) p_{J_\eta,1}(h_\eta - g_\eta). \tag{28}$$

Let

$$d_{\varepsilon,\eta,\rho} = P_{K,0}(w_{\varepsilon,\eta,\rho}) \left(1 + P_{K,0}(u_{\varepsilon,\eta,\rho}) M_1'\right) + P_{K,0}(u_{\varepsilon,\eta,\rho}) p_{J_\eta,1}(h_\eta - g_\eta).$$

We have $\left(p_{J_\eta,1}(h_\eta - f_\eta)\right)_\eta \in I_A$ and $(w_{\varepsilon,\eta,\rho})_{\varepsilon,\eta,\rho} \in \mathcal{J}(\mathbb{R}^2)$, then

$$(d_{\varepsilon,\eta,\rho})_\eta \in |I_A|$$

$(P2)$ $\quad F(t,x,\sigma_\eta(v_{\varepsilon,\eta,\rho}(t,x))) - F(t,x,\phi_\eta(u_{\varepsilon,\eta,\rho}(t,x)))$

$$= \Delta_{\varepsilon,\eta,\rho}(t,x) \left(\int_0^1 \frac{\partial F}{\partial z}(t,x,\phi_\eta(u_{\varepsilon,\eta,\rho}(t,x))) + \xi(\sigma_\eta(v_{\varepsilon,\eta,\rho}(t,x)) - \phi_\eta(u_{\varepsilon,\eta,\rho}(t,x)))\,\mathrm{d}\xi\right),$$

Let $(t,x) \in K$, we have

$$\begin{aligned}|F(t,x,\sigma_\eta(v_{\varepsilon,\eta,\rho}(t,x))) - F(t,x,\phi_\eta(u_{\varepsilon,\eta,\rho}(t,x)))| &\leq d_1 r_\eta^{p_0} |\Delta_{\varepsilon,\eta,\rho}(t,x)| \\ &\leq d_1 r_\eta^{p_0} d_{\varepsilon,\eta,\rho}.\end{aligned}$$

We deduce that

$$\left(p_{K,0}\left(F(\cdot,\cdot,\sigma_\eta(v_{\varepsilon,\eta,\rho})) - F(\cdot,\cdot,\phi_\eta(u_{\varepsilon,\eta,\rho}))\right)\right)_{\varepsilon,\eta,\rho} \in |I_A|.$$

This implies the 0th order estimate. According to Proposition 2

$$\left(F(\cdot,\cdot,\sigma_\eta(v_{\varepsilon,\eta,\rho})) - F(\cdot,\cdot,\phi_\eta(u_{\varepsilon,\eta,\rho}))\right)_{\varepsilon,\eta,\rho} \in \mathcal{J}(\mathbb{R}^2).$$

■

4.4.2 Dependence of the generalized solution from the class $[l_\varepsilon]$

We need the following

Lemma 19 *Let $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$ such that for every ε, $f_\varepsilon, g_\varepsilon$ are bijective and $\left(f_\varepsilon^{-1}\right)_\varepsilon, \left(g_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$. If moreover $(g_\varepsilon - f_\varepsilon)_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$ we have that*

$$\left(f_\varepsilon^{-1} - g_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$$

The proof will use the pointvalues characterization; so let us first define the following map (cf [15], 1.2)

$$\begin{aligned}\Theta : \mathcal{G}_\tau(\mathbb{R}) &\to \mathcal{F}(\overline{\mathbb{R}}) \\ [f_\varepsilon] &\mapsto f : \tilde{x} = [x_\varepsilon] \mapsto f(\tilde{x}) = [f_\varepsilon(x_\varepsilon)]\end{aligned}$$

where $\overline{\mathbb{R}}$ denotes the field of generalized real numbers and $(\overline{\mathbb{R}})$ is the set of map from $\overline{\mathbb{R}}$ to $\overline{\mathbb{R}}$.

Proof. First $\mathcal{G}_\tau(\mathbb{R})$ and $\mathcal{F}(\overline{\mathbb{R}})$ can be endowed with a structure of unitary rings where the operations are addition and composition of functions (the unit is then the identity function). Let us prove that Θ is a morphism between these rings; let $(f_\varepsilon)_\varepsilon, (g_\varepsilon)_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$ and $f = \Theta([f_\varepsilon])$, $g = \Theta([g_\varepsilon])$, $h = \Theta([f_\varepsilon \circ g_\varepsilon])$ (note that $(f_\varepsilon \circ g_\varepsilon)_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$):

$$\begin{aligned}\forall \tilde{x} = [x_\varepsilon] \in \overline{\mathbb{R}}, h(\tilde{x}) = [(f_\varepsilon \circ g_\varepsilon)(x_\varepsilon)] &= [f_\varepsilon(g_\varepsilon(x_\varepsilon))] \\ &= f(g(\tilde{x})) \\ &= \Theta([f_\varepsilon]) \circ \Theta([g_\varepsilon])(\tilde{x})\end{aligned}$$

If we assume moreover that $f_\varepsilon, g_\varepsilon$ are bijective, $\left(f_\varepsilon^{-1}\right)_\varepsilon, \left(g_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{X}_\tau(\mathbb{R})$ and $(g_\varepsilon - f_\varepsilon)_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$, we have that:

$$\begin{aligned}Id = [f_\varepsilon \circ f_\varepsilon^{-1}] = \Theta([f_\varepsilon]) \circ \Theta([f_\varepsilon^{-1}]) \\ Id = [f_\varepsilon^{-1} \circ f_\varepsilon] = \Theta([f_\varepsilon^{-1}]) \circ \Theta([f_\varepsilon])\end{aligned}$$

So that $\Theta([f_\varepsilon^{-1}]) = \Theta([f_\varepsilon])^{-1}$. Now as $[g_\varepsilon] = [f_\varepsilon]$, we have that $f = g$ so that $f^{-1} = g^{-1}$ and then $[f_\varepsilon^{-1}] = [g_\varepsilon^{-1}]$ which concludes the lemma. ∎

Theorem 20 *With the previous hypotheses, the generalized solution $u = [u_{\varepsilon,\eta,\rho}]$ of the characteristic Cauchy problem (P_g) and, a fortiori, any solution of it depends solely on l ($l \in \mathcal{X}_\tau(\mathbb{R})$) as generalized functions and not on their representatives $(l_\varepsilon)_\varepsilon$.*

Proof. Take K a compact subset of $\mathbb{R}^2$. Consider the compact subset K_ε built as in Section 4.2. We will prove that

$$\forall \alpha \in \mathbb{N}^2, P_{K_\varepsilon,k}(w_{\varepsilon,\eta,\rho}) \in |I_A|,$$

where $w_{\varepsilon,\eta,\rho} = (v_{\varepsilon,\eta,\rho} - u_{\varepsilon,\eta,\rho})$,

$$u_{\varepsilon,\eta,\rho}(t,x) = f_\rho(l_\varepsilon^{-1}(x)) + \int_{l_\varepsilon^{-1}(x)}^t F_\eta(\tau, x, u_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau, \tag{29}$$

$$v_{\varepsilon,\eta,\rho}(t,x) = f_\rho(p_\varepsilon^{-1}(x)) + \int_{p_\varepsilon^{-1}(x)}^t F_\eta(\tau, x, v_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau. \tag{30}$$

$(H'5)$ $\quad l_\varepsilon, p_\varepsilon \in \mathrm{C}^\infty(\mathbb{R})$, $l_\varepsilon, p_\varepsilon$ strictly increasing, $l_\varepsilon(\mathbb{R}) = \mathbb{R}$, $p_\varepsilon(\mathbb{R}) = \mathbb{R}$.

$(H'6)$

$$(l_\varepsilon)_\varepsilon, \left(l_\varepsilon^{-1}\right)_\varepsilon, (p_\varepsilon)_\varepsilon, \left(p_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{X}_\tau(\mathbb{R}), \quad (l_\varepsilon)_\varepsilon, (p_\varepsilon)_\varepsilon \text{ c-bounded and } \lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\mathbb{R})}} l_\varepsilon = 0, \lim_{\substack{\varepsilon \to 0 \\ \mathcal{D}'(\mathbb{R})}} p_\varepsilon = 0.$$

such that $(p_\varepsilon - l_\varepsilon)_\varepsilon \in \mathcal{N}_\tau(\mathbb{R})$. Moreover we have

$$\left(l_\varepsilon^{-1} - p_\varepsilon^{-1}\right)_\varepsilon \in \mathcal{J}(\mathbb{R}).$$

We compute

$$
\begin{aligned}
w_{\varepsilon,\eta,\rho}(t,x) &= f_\rho(p_\varepsilon^{-1}(x)) - f_\rho(l_\varepsilon^{-1}(x)) \\
&\quad + \int_{p_\varepsilon^{-1}(x)}^{t} F_\eta(\tau,x,v_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau - \int_{l_\varepsilon^{-1}(x)}^{t} F_\eta(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau.
\end{aligned}
$$

then

$$
\begin{aligned}
&\int_{p_\varepsilon^{-1}(x)}^{t} F_\eta(\tau,x,v_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau - \int_{l_\varepsilon^{-1}(x)}^{t} F_\eta(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau \\
&= \int_{p_\varepsilon^{-1}(x)}^{t} \left(F_\eta(\tau,x,v_{\varepsilon,\eta,\rho}(\tau,x)) - F_\eta(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x))\right)\,\mathrm{d}\tau \\
&+ \int_{p_\varepsilon^{-1}(x)}^{t} F_\eta(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau - \int_{l_\varepsilon^{-1}(x)}^{t} F_\eta(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau
\end{aligned}
$$

thus

$$
\begin{aligned}
&\int_{p_\varepsilon^{-1}(x)}^{t} F_\eta(\tau,x,v_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau - \int_{l_\varepsilon^{-1}(x)}^{t} F_\eta(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau \\
&= \int_{p_\varepsilon^{-1}(x)}^{t} w_{\varepsilon,\eta,\rho}(\tau,x)\left(\int_0^1 \frac{\partial F_\eta}{\partial z}(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x)+\theta w_{\varepsilon,\eta,\rho}((\tau,x))\,\mathrm{d}\theta\right)\,\mathrm{d}\tau \\
&+ \int_{p_\varepsilon^{-1}(x)}^{l_\varepsilon^{-1}(x)} F_\eta(\tau,x,u_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau.
\end{aligned}
$$

As $\sup\limits_{(t,x,z)\in K_\varepsilon\times\mathbb{R}} \left|\dfrac{\partial F_\eta}{\partial z}(t,x,z)\right| = M_{K_\varepsilon,\eta,1} \leq \mu_{K,1} M_{\varepsilon,\eta}$,We deduce that

$$
\begin{aligned}
|w_{\varepsilon,\eta,\rho}(t,x)| &\leq \sup_{K_{1\varepsilon}} \left|f_\rho(p_\varepsilon^{-1}(x)) - f_\rho(l_\varepsilon^{-1}(x))\right| + \mu_{K,1} M_{\varepsilon,\eta} \int_{l_\varepsilon^{-1}(x)}^{t} |w_{\varepsilon,\eta,\rho}(\tau,x)|\,\mathrm{d}\tau \\
&\quad + \left|\int_{p_\varepsilon^{-1}(x)}^{l_\varepsilon^{-1}(x)} P_{K_{\lambda,\varepsilon},0}(F_\eta(\cdot,\cdot,u_{\varepsilon,\eta,\rho})\,\mathrm{d}\tau\right|.
\end{aligned}
$$

According to (11), we have

$$
P_{K_\varepsilon,0}(F_\eta(\cdot,\cdot,u_{\varepsilon,\eta,\rho})) \leq c_{3\varepsilon\eta} + c_{4\varepsilon\eta} P_{K_{1\varepsilon},0}(f_\rho),
$$

we deduce that

$$
\begin{aligned}
|w_{\varepsilon,\eta,\rho}(t,x)| &\leq \sup_{K_{1\varepsilon}} \left|f_\rho(p_\varepsilon^{-1}(x)) - f_\rho(l_\varepsilon^{-1}(x))\right| + \mu_{K,1} M_{\varepsilon,\eta} \int_{l_\varepsilon^{-1}(x)}^{t} |w_{\varepsilon,\eta,\rho}(\tau,x)|\,\mathrm{d}\tau \\
&\quad + \left\|l_\varepsilon^{-1} - p_\varepsilon^{-1}\right\|_{K_{1\varepsilon}} \left(c_{3\varepsilon\eta} + c_{4\varepsilon\eta} P_{K_{1\varepsilon},0}(f_\rho)\right).
\end{aligned}
$$

Thus

$$
\begin{aligned}
|w_{\varepsilon,\eta,\rho}(t,x)| &\leq \left\|l_\varepsilon^{-1} - p_\varepsilon^{-1}\right\|_{K_{1\varepsilon}} P_{K_{1\varepsilon},1}(f_\rho) + \mu_{K,1} M_{\varepsilon,\eta} \int_{l_\varepsilon^{-1}(x)}^{t} |w_{\varepsilon,\eta,\rho}(\tau,x)|\,\mathrm{d}\tau \\
&\quad + \left\|l_\varepsilon^{-1} - p_\varepsilon^{-1}\right\|_{K_{1\varepsilon}} \left(c_{3\varepsilon\eta} + c_{4\varepsilon\eta} P_{K_{1\varepsilon},0}(f_\rho)\right) \\
&\leq \left\|l_\varepsilon^{-1} - p_\varepsilon^{-1}\right\|_{K_{1\varepsilon}} \Psi_{I_{\lambda,\varepsilon},0}(f_\rho) + \mu_{K,1} M_{\varepsilon,\eta} \int_{l_\varepsilon^{-1}(x)}^{t} |w_{\varepsilon,\eta,\rho}(\tau,x)|\,\mathrm{d}\tau,
\end{aligned}
$$

with $\Psi_{K_{1\varepsilon}}(f_\rho) = (c_{3\varepsilon\eta} + c_{4\varepsilon\eta} P_{K_{1\varepsilon},0}(f_\rho) + P_{K_{1\varepsilon},1}(f_\rho))$. Put $\varepsilon(\tau) = \sup_{x\in K_{2\varepsilon}} |w_{\varepsilon,\eta,\rho}(\tau,x)|$. Then

$$|w_{\varepsilon,\eta,\rho}(t,x)| \leq \mu_{K,1} M_{\varepsilon,\eta} \int_{l_\varepsilon^{-1}(x)}^{t} e(\tau)\,\mathrm{d}\tau + \left\| l_\varepsilon^{-1} - p_\varepsilon^{-1} \right\|_{K_{1\varepsilon}} \Psi_{K_{1\varepsilon}}(f_\rho),$$

We deduce that

$$\forall t \in K_{1\varepsilon},\, e(t) \leq \mu_{K,1} M_{\varepsilon,\eta} \int_{l_\varepsilon^{-1}(x)}^{t} e(\tau)\,\mathrm{d}\tau + \left\| l_\varepsilon^{-1} - p_\varepsilon^{-1} \right\|_{K_{1\varepsilon}} \Psi_{K_{1\varepsilon}}(f_\rho).$$

Thus, according to Gronwall's lemma,

$$\forall t \in K_{1\varepsilon},\, e(t) \leq \exp\left(\int_{l_\varepsilon^{-1}(x)}^{t} \mu_{K,1} M_{\varepsilon,\eta}\,\mathrm{d}\tau \right) \left\| l_\varepsilon^{-1} - p_\varepsilon^{-1} \right\|_{K_{1\varepsilon}} \Psi_{K_{1\varepsilon}}(f_\rho),$$

then

$$e(t) \leq \exp(\left(l_\varepsilon^{-1}(b) - l_\varepsilon^{-1}(-b)\right) \mu_{K,1} M_{\varepsilon,\eta}) \left\| l_\varepsilon^{-1} - p_\varepsilon^{-1} \right\|_{I_\lambda} \Psi_{K_{1\varepsilon}}(f_\rho)$$

and consequently

$$P_{K_{\lambda,\varepsilon},0}(w_{\varepsilon,\eta,\rho}) \leq \exp(\left(l_\varepsilon^{-1}(b) - l_\varepsilon^{-1}(-b)\right) \mu_{K,1} M_{\varepsilon,\eta}) \left\| l_\varepsilon^{-1} - p_\varepsilon^{-1} \right\|_{K_{1\varepsilon}} \Psi_{K_{1\varepsilon}}(f_\rho).$$

Then $P_{K_{\lambda,\varepsilon},0}(w_{\varepsilon,\eta,\rho}) \in |I_A|$, this implies the 0^{th} order estimate. The hypotheses of Proposition 2 are fulfilled. Indeed the set B is stable by inverse and contains the element $(\rho)_{\varepsilon,\eta,\rho}$ such that $\lim_\Lambda \rho = 0$. It follows that $(p_{K,l}(w_{\varepsilon,\eta,\rho}))_{\varepsilon,\eta,\rho} \in |I_A|$ for any $l \in \mathbb{N}$. we deduce $(w_{\varepsilon,\eta,\rho})_\varepsilon \in \mathcal{J}(\mathbb{R}^2)$; consequently u depends solely on the class $[l_\varepsilon]$ as a generalized function, not on the particular representative. ■

4.5 Comparison with classical solutions in non characteristic case

Even if the data are as irregular as distributions, it may happen that the initial formal ill-posed problem (P_{form}) has nonetheless a local smooth solution as it will be seen in the example 4.7.2. We are going to prove that this solution is exactly the restriction (according to the sheaf theory sense) of the generalized one.

The generalized solution to Problem (P_g) is defined from the integral representation (9). Thus, we are going to study the relationship between this generalized function and the classical solutions to (P_{form}) (when they exist) on a domain Ω such that $\Omega = \left]-\mu,\mu\right[\times \left]-\nu,\nu\right[$ when $(\mu,\nu) \in \mathbb{R}_+^2$.

Let γ be a non characteristic curve of equation $x = l(t)$. Remark that, in this case, the families $(\rho)_{\eta,\rho}$ and $(\exp \dfrac{1}{\eta^2})_{\eta,\rho}$ are sufficient to overgenerate $\mathcal{C}$.

Recall that there exists a canonical sheaf embedding of $\mathrm{C}^\infty(\cdot)$ into $\mathcal{A}(\cdot)$, through the morphism of algebra

$$\sigma_O : \mathrm{C}^\infty(O) \to \mathcal{A}(O),\quad f \mapsto [f_{\eta,\rho}] \quad (\text{where } O \text{ is any open subset of } \mathbb{R}^2 \text{and } f_{\eta,\rho} = f).$$

The presheaf $\mathcal{A}$ allows to restriction and as usually we denote by $u|_O$ the restriction on O of $u \in \mathcal{A}\left(\mathbb{R}^2\right)$.

Theorem 21 *Let $u = [u_{\eta,\rho}]$ be the generalized solution $u \in \mathcal{A}\left(\mathbb{R}^2\right)$ to Problem (P_g), given by Proposition 11. Let -$\Omega = \left]-\mu,\mu\right[\times \left]-\nu,\nu\right[$ be an open box of $\mathbb{R}^2$. Assume that Problem (P_{form}) admits a smooth solution v on Ω and that $\Omega = \bigcup_\eta \Omega_\eta$ where $(\Omega_\eta)_\eta$ is an increasing family of open boxes of $\mathbb{R}^2$ such that $\sup_{(x,y)\in\Omega_\eta} |v(x,y)| < r_\eta - 1$ for any η. Then v (element of $\mathrm{C}^\infty(\Omega)$ canonically embedded in $\mathcal{A}(\Omega)$) is the restriction (according to the sheaf theory sense) of u to Ω, $v = u|_\Omega$.*

Proof. We clearly have

$$\forall (x,t) \in \Omega, \quad v(x,t) = f_\rho(l^{-1}(x)) + \int_{l^{-1}(x)}^{t} F(\tau, x, v(\tau, x))\,\mathrm{d}\tau,$$

as $[0,T] \subset \left]-\mu, \mu\right[$. We take as representative of u the family $(u_{\eta,\rho})_{\eta,\rho}$ given by Proposition 11. This family satisfies

$$\forall (t,x) \in \Omega, \quad u_{\eta,\rho}(t,x) = f_\rho(l^{-1}(x)) + \int_{l^{-1}(x)}^{t} F_\eta(\tau, x, u_{\varepsilon,\eta,\rho}(\tau, x))\,\mathrm{d}\tau.$$

Set $(w_{\eta,\rho})_{\eta,\rho} = \left(u_{\eta,\rho}|_\Omega - v\right)_{\eta,\rho}$ and take $K \Subset \Omega$. As Ω is a box, there exists $\lambda > 0$ such that $K \subset [-\lambda, \lambda] \times [-\lambda', \lambda'] \subset \Omega$. Moreover, there exists η_0 such that, for all $\eta \leq \eta_0$, $K_\lambda = [-\lambda, \lambda] \times [-\lambda', \lambda'] \Subset \Omega_\eta$ as Ω_η is also a box. Note that, for $(\tau, x, z) \in \Omega_\eta \times \left]-r_\eta + 1, r_\eta - 1\right[$, we have $F(\tau, x, z) = F_\eta(\tau, x, z)$ by construction of F_η.

Thus v, which values are in $\left]-r_\eta + 1, r_\eta - 1\right[$, is solution of the same integral equation as $u_{\eta,\rho}$, which admits a unique solution since F_η is a smooth function of its arguments. Thus, for all $\eta \leq \eta_0$, v and $u_{\eta,\rho}$ are equal on Ω_η. Then $\left(P_{K,n}(v)\right)_{\varepsilon,\eta,\rho} \in A$ for any $K \Subset \Omega$ and $n \in \mathbb{N}$. Then v (identified with $[(v)_{\eta,\rho}]$) belongs to $\mathcal{A}(\Omega)$. Moreover, for all $\eta \leq \eta_0$, $\sup_{(x,y)\in K_\lambda} |w_{\eta,\rho}| = 0$, hence $(P_{K,l}(w_{\eta,\rho}))_{\eta,\rho} \in |I_A|$ for any $l \in \mathbb{N}$ as $w_{\eta,\rho}$ vanishes on K. Thus $(w_{\eta,\rho})_{\eta,\rho} \in \mathcal{N}(\Omega)$ and $v = u|_\Omega$ as claimed. ■

Remark 7 *The hypotheses made in the previous theorem are satisfied for the example $F(.,.,u) = u^k$ $(k \geq 2)$ and $f = 1$, for which the local solution $v(t,x) = (1-(k-1)t)^{-(k-1)^{-1}}$ exists in $\left]-\infty, (k-1)^{-1}\right[\times \mathbb{R}$. On $\Omega_\eta = \left]-\infty, (1-\eta)(k-1)^{-1}\right[\times \mathbb{R}$, we have $|v(t,x)| \leq \eta^{-(k-1)^{-1}}$. It suffices to take $r_\eta > \eta^{-(k-1)^{-1}}$, say $r_\eta = \eta^{-1}$.*

4.6 Continuous dependence from the data

For linear (or semi linear problems) with irregular data, we refer to [8, 10] in which this continuous dependence is shown for Colombeau type algebras. For problems with other type of singularities, which cannot be handled in a one parametrized algebras, a first step is to extend sharp topologies and functorial properties to the case of $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebra, which is done in [6]. The further step is to show that the following Hadamard setting will hold.

Proposition 22 *Let $u(h,\mathcal{F},\mathcal{R})$ be a solution to the generalized problem*

$$\frac{\partial u}{\partial t} = \mathcal{F}(u) \;; \quad \mathcal{R}(u) = h$$

with $h \in \mathcal{A}(\mathbb{R})$. Then, at least in a neighborhood of $\mathbf{f}$ *(see Proposition 8), the map*

$$\mathcal{A}(\mathbb{R}) \to \mathcal{A}\left(\mathbb{R}^2\right), \quad h \mapsto u(h,\mathcal{F},\mathcal{R})$$

is continuous for the corresponding sharp topologies.

Proof. From the proof of Theorem 11, it follows that

$$P_{K_\varepsilon,l}(u_{\varepsilon,\eta,\rho}) \leq c_{\varepsilon,l,0} + \sum_{j=1}^{l} c_{\varepsilon,l,j}\left(p_{K_{1\varepsilon},j}(f_\rho)\right)^{\alpha_j}$$

where coefficients $c_{\varepsilon,l,j}$ belong to $\mathcal{C}$. Thus we have the result. ■

4.7 Examples

The previous results show that the solution of (P_g) is the class in $\mathcal{A}\left(\mathbb{R}^2\right)$ of the solution of (P_∞) given in Proposition 9 and verifying

$$u_{\varepsilon,\eta,\rho}(t,x) = f_\rho(l_\varepsilon^{-1}(x)) + \int_{l_\varepsilon^{-1}(x)}^{t} F_\eta(\tau, x, u_{\varepsilon,\eta,\rho}(\tau,x))\,\mathrm{d}\tau.$$

It depends on the class $[g_\eta]$ of cut-off functions and on the class $[l_\varepsilon]$.

We can study some special cases of this result when studying isolated singularities and simplifying the regularizations or the problem (P_c).

4.7.1 A purely characteristic case

To simplify the problem, suppose $F = 0$ and set

$$(P_{char}) \begin{cases} \dfrac{\partial u}{\partial t} = 0 \\ u\mid_{\{t=0\}} = f \end{cases}$$

where $f \in \mathrm{C}^\infty(\mathbb{R})$. As it is mentioned previously we regularize (P_{char}) by choosing $l_\varepsilon(t) = \varepsilon t$ in

$$(P_\infty) \begin{cases} \dfrac{\partial}{\partial t}\left(u_\varepsilon(t,x)\right) = 0 \\ u_\varepsilon(t,\varepsilon t) = f(t) \end{cases}$$

the parameters η and ρ being fixed. Clearly the solution of (P_∞) is

$$u_\varepsilon(t,x) = f\left(\frac{x}{\varepsilon}\right)$$

and the generalized solution u of (P_g) is the class in $\mathcal{A}\left(\mathbb{R}^2\right)$ of the map $(t,x) \mapsto f(x/\varepsilon)$. Remark that here $\mathcal{C}$ is overgenerated by the family $(\varepsilon)_\varepsilon$, that is to say $\mathcal{A}\left(\mathbb{R}^2\right)$ is exactly the simplified Colombeau algebra $\mathcal{G}\left(\mathbb{R}^2\right)$.

There is no classical object corresponding to that generalized function. However we can show it is possible to link the generalized solution u to a distribution by means of an association process defined in subsection 2.2. Suppose that f is integrable with $\int f(x)\,\mathrm{d}x = 1$ and write

$$\frac{1}{\varepsilon}u_\varepsilon = (t,x) \mapsto 1_t \otimes \frac{1}{\varepsilon} f\left(\frac{x}{\varepsilon}\right).$$

We have clearly

$$\lim_{\substack{D'(\mathbb{R}^2)\\ \varepsilon\to 0}} \left(\frac{1}{\varepsilon}u_\varepsilon\right) = 1_t \otimes \delta_x = \delta_\Gamma$$

where δ_Γ is the Dirac distribution on the characteristic manifold $\Gamma = \left\{(t,x) \in \mathbb{R}^2 : t = 0\right\}$. This leads to the

Proposition 23 *With the previous hypotheses, the solution u of the generalized problem (P_g) associated to (P_{char}) satisfies*

$$u \underset{\varepsilon}{\sim} \delta_\Gamma.$$

In other words, u have a bidimensional soliton structure, and $\operatorname{supp} u = \operatorname{supp} \delta_\Gamma = \Gamma$: the solution of this characteristic Cauchy problem is associated to a bidimensional soliton whose support is the characteristic curve.

4.7.2 A purely non Lipschizian case

We start from the simplified case where ρ and ε are constant, for example

$$(P_{nonLip}) \begin{cases} \dfrac{\partial u}{\partial t} = u^2 \\ u(t,t) = f(t) \end{cases}$$

which admits a local solution as

$$v\left(t,x\right) = \frac{f\left(x\right)}{1+\left(x-t\right)f\left(x\right)}$$

If we choose $f\left(t\right) = 1$ we have a simplified local solution

$$v\left(t,x\right) = \frac{1}{1+\left(x-t\right)}$$

on $\Omega = \{(x,t) : x > t-1\}$.

Let be (P_{gen}) the generalized associated problem

$$P_{gen} \begin{cases} \dfrac{\partial u}{\partial t} = \mathcal{F}(u), \\ u|_{\gamma} = \mathbf{f} \end{cases}$$

where γ is the curve of equation $x = t$, $\mathbf{f} = f$ and $\mathcal{F}$ is associated to $F = u^2$ via the family $\left(g_\eta\right)_\eta$ given in (4.1.1). To solve Problem (P_{gen}) we can consider the family of problems

$$(P_\eta) \begin{cases} \dfrac{\partial u_\eta}{\partial t} = \left(u_\eta\left(x,t\right)\right)g_\eta(u_\eta\left(x,t\right)))^2, \\ u_\eta(t,t) = f(t) \end{cases}$$

If u_η is a solution to (P_η) then $u = [u_\eta]$ *is solution to* (P_{gen}). Theorem 21 shows that the restriction of $u \in \mathcal{A}\left(\mathbb{R}^2\right)$ to Ω is precisely v. The local classical solution v which blows-up for $x = t-1$, extends to a global generalized solution u which absorbs this blow-up.

References

[1] E. Allaud, V. Dévoué. *Generalized solutions to a characteristic Cauchy problem.* CEREGMIA working paper 2010-03, (www.ceregmia.eu).

[2] J.-F. Colombeau. *New Generalized Functions and Multiplication of Distributions.* North-Holland, Amsterdam, Oxford, New-York, 1984.

[3] J.-F. Colombeau. *Multiplication of Distributions: a tool in mathematics, numerical engineering and theoretical physics.* Lecture Notes in Mathematics 1532, Springer-Verlag, Berlin, 1992

[4] G. M. Constantine, H. Savits. *A multivariate Faa di Bruno formula with applications.* Trans. Amer. Math. Soc. **348** (2) (1996), 503–520.

[5] A. Delcroix. *Remarks on the embedding of spaces of distributions into spaces of Colombeau generalized functions.* Novisad J. math. **35**/2 (2005), 27–40.

[6] A. Delcroix. *Topology and functoriality in $(\mathcal{C}, \mathcal{E}, \mathcal{P})$-algebras. Application to singular differential problems.* J. Math. Anal. Appl. **359** (2008), 394–403, Doi:10.1016/j.jmaa.2009.05.046

[7] A. Delcroix. *A new approach to temperate generalized Colombeau functions.* Publ. Inst. Math. (Beograd) (N.S.) **84**(98) (2008), 109–121, Doi: 10.2298/PIM0898109D

[8] A. Delcroix, V. Dévoué, J.-A. Marti. *Generalized solutions of singular differential problems. Relationship with classical solutions.* J. Math. Anal. Appl. **353** (2009), 386–402. Doi:10.1016/j.jmaa.2008.11.077J. Math. Anal. Appl. **353** (2009), 386–402. Doi:10.1016/j.jmaa.2008.11.077

[9] A. Delcroix, J.-A. Marti, M. Oberguggenberger. *Spectral asymptotic analysis in algebras of generalized functions.* Asymptot. Anal. **59**(1-2) (2008), 83–107

[10] A. Delcroix, D. Scarpalézos. *Topology on Asymptotic Algebras of Generalized Functions and Applications.* Monatsh. Math. **129** (2000), 1–14.

[11] V. Dévoué. *On generalized solutions to the wave equation in canonical form.* Dissertationes math. **443** (2007), 1–69.

[12] V. Dévoué. *Generalized solutions to a non Lipschitz Cauchy problem.* J. Appl. Anal. Vol. 15, No. 1 (2009), 1–32.

[13] V. Dévoué. *Generalized solutions to a non Lipschitz Goursat problem.* Diff. Equa. & Appl. Vol. 1, No. 2 (2009), 153–178.

[14] I. Gelfand, G. Shilov. *Generalized Functions.* Vol. 1 & 2, Academic Press, 1964 & 1968.

[15] M. Grosser, M. Kunzinger, M. Oberguggenberger, R. Steinbauer. *Geometric Theory of Generalized Functions with Applications to General Relativity.* Kluwer Academic Publ., Dordrecht (2001).

[16] S. Haller, G. Hörmann. *Comparison of some solution concepts for linear first-order hyperbolic differential equations with non-smooth coefficients.* Publ. Inst. Math. (Beograd) (N.S.) **84**(98) (2008), 123–157. Doi: 10.2298/PIM0898123H

[17] M. Hasler, J.-A. Marti. *Functorial methods in asymptotic extensions of topological algebras.* Preprint (2009).

[18] G. Hörmann, M.-V. De Hoop. *Microlocal analysis and global solutions for some hyperbolic equations with discontinuous coefficients.* Acta Appl. Math. **67** (2001), 173–224.

[19] J.-A. Marti. *$(\mathcal{C}, \mathcal{E}, \mathcal{P})$-Sheaf structure and applications.* In: *Nonlinear theory of generalized functions.* M. Grosser and alii, Eds. Research notes in mathematics. Chapman & Hall/CRC (1999), 175–186.

[20] J.-A. Marti. *Non linear Algebraic analysis of delta shock wave to Burgers' equation.* Pacific J. Math. **210**(1) (2003), 165–187.

[21] J.-A. Marti, P. S. Nuiro, V. S. Valmorin. *A nonlinear Goursat problem with irregular data.* Integral Transforms Spec. Funct. **6**(1-4) (1998), 229–246.

[22] M. Nedeljkov, S. Pilipović, D. Scarpalezos. *The linear theory of Colombeau generalized functions.* Pitman Research Notes in Mathematics Series 385, Longman Scientific & Technical, Harlow (1998).

[23] M. Oberguggenberger. *Multiplication of Distributions and Applications to Partial Differential Equations.* Pitman Research Notes in Mathematics 259, Longman Scientific & Technical, Harlow (1992).

[24] M. Oberguggenberger. *Generalized solutions to nonlinear wave equations.* Matemática Contemporânea **27** (2004), 169–187.

[25] S. Pilipovic, D. Scarpalézos. *Divergent type quasilinear Dirichlet problem.* Acta Appl. Math. **94** (2006) 67–82.

[26] D. Pandres. *On higher ordered differentiation.* Am. Math. Monthly. **64,** (8) (1957), 566–572.

[27] D. Scarpalézos. *Colombeau's generalized functions: Topological structures; Microlocal properties. A simplified point of view. Part I.* Bull. Cl. Sci. Math. Nat. Sci. Math. **25** (2000), 89–114.

[28] D. Scarpalézos. *Colombeau's generalized functions: Topological structures; Microlocal properties. A simplified point of view. Part II.* Publ. Inst. Math. (Beograd) (N.S.) **76**(90) (2004), 111–125.

[29] K. Travers. *Semilinear hyperbolic systems in one space dimension with strongly singular initial data.* Electron. J. Diff. Eqns. 1997(14) (1997), 1–11.

[30] H. Vernaeve. *Pointwise Characterization in generalized function algebras.* Preprint. *http://arxiv.org/abs/0712.3464.*